Homework #3
Nonlinear dynamics and chaos

1. **Linearized 2d systems:** Classify the stability of the fixed points of the following systems by solving for their eigenvalues/ vectors and plotting the vector field in the phase plane. Include several typical trajectories. Can use the quiver function of Matlab for the plot.

$$\dot{x} = x - y; \qquad \dot{y} = x + y \tag{1}$$
$$\dot{x} = 5x + 2y; \qquad \dot{y} = -17x - 5y \tag{2}$$

2. **Nonlinear 2d systems:** Find the fixed points, classify them, sketch neighboring trajectories, and try to fill in the rest of the phase space portrait:

$$\dot{x} = xy - 1; \qquad \dot{y} = x - y^3 \tag{3}$$
$$\dot{x} = xy; \qquad \dot{y} = x^2 - y \qquad \text{beware, linearization fails here. why?} \tag{4}$$

3. **Phase locking:** Consider the circle map:

$$\theta_{n+1} = \theta_n + \Omega - \frac{K}{2\pi}\sin(2\pi\theta_n) \qquad mod\ 1 \tag{5}$$

a Matlab program for the circle map is on the course home page.

(a) Find a period 2 solution ($p/q = 1/2$) for $K > 1/2$, $\Omega \neq 1/2$.

i. Show that the solution does not depend on the initial conditions θ_0 by iterating the map from 2 different initial conditions, converging to the same period-2 solution.

ii. Find another ($p/q = 1/2$) solution for different K, Ω, with again $K > 1/2$, $\Omega \neq 1/2$, show that it is different from the previous one although it has the same period

(b) Find a $p/q = 0/1$ solution for $\Omega \neq 0$, $K > 1/2$. Describe the behavior of this solution as function of n. Do the same for a $p/q = 1/1$ solution for $\Omega \neq 1$, $K > 1/2$.

(c) find a solution $p/q = 3/4$ for $K > 1/2$.

(d) try $K > 1$. What happens? (chaos for large enough $K\ldots$)

4. **Optional challenge problem (interesting!): Analytic calculation of Arnold's tongues in the circle map near $K = 0$:**

   (a) Consider the circle map

   $$x_{n+1} = F(x_n) = x_n + \Omega - \frac{K}{2\pi}\sin(2\pi x_n) \qquad \text{(mod 1)}.$$

   It can be shown that the rotation (winding) number is p/q if and only if

   $$F^q(x) - (x + p) = 0.$$

   First, test this numerically for some two different values of (p, q). Next, use this relation to show that the edges of the Arnold tongues $q = 1$ and $p = 0$ or $p = 1$ are at $\Omega = K/(2\pi)$ (for $p = 0$) and $\Omega = 1 - K/(2\pi)$ (for $p = 1$).

   (b) Using the same approach as the previous question, show that the boundary of the Arnold tongues $p = 1$ and $q = 2$ for small K is given by $\Omega = \frac{1}{2} \pm \frac{K^2}{8\pi}$.