Problem #1

Standard Map: 

\[ \theta_{n+1} = \theta_n + \rho_n \mod 2\pi \]
\[ p_{n+1} = p_n + k \sin \theta_{n+1} \]

To find fixed points \((\theta, p)\) solve equations:

(i) \[ \theta = \theta + p \mod 2\pi \]
(ii) \[ p = p + k \sin \theta \quad (-\pi < p < \pi) \]

(i) is satisfied by \( p = 0 \) in the interval \( |p| < \pi \).

For \( k = 0 \), (ii) is trivially satisfied for any \( \theta \).

If \( k > 0 \), (ii) \( \Rightarrow \theta = 0 \) or \( \pi \).

Thus, fixed points are:

\[
\begin{align*}
K = 0: & \quad (\theta, p) = (\alpha, 0) \text{ any } \alpha. \\
K > 0: & \quad (\theta, p) = (0, 0) \text{ or } (\pi, 0)
\end{align*}
\]
To complete the study of $f_1(\rho, \theta)$, we turn to the Jacobian:

$$
J = \begin{pmatrix}
\frac{\partial f_{11}}{\partial \rho} & \frac{\partial f_{11}}{\partial \theta} \\
\frac{\partial f_{12}}{\partial \rho} & \frac{\partial f_{12}}{\partial \theta}
\end{pmatrix} = \begin{pmatrix}
1 & 1 \\
K \cos(\theta + \rho) & 1 + K \cos(\theta + \rho)
\end{pmatrix}
$$

Thus, $\text{Trace}(J) = 2 + K \cos(\theta + \rho)$.

Investigate fixed points for $K \neq 0$:

$$(0, 0): \quad \text{Trace}(J) = 2 + K$$

Thus, $(0, 0)$ is hyperbolic for $K > 0$, hyperbolic with reflection for $0 < K < 0$, and elliptic for $-4 < K < 0$.

$$(\pi, 0): \quad \text{Trace}(J) = 2 - K$$

Similarly, $(\pi, 0)$ is hyperbolic for $K < 0$, hyperbolic with reflection for $K > 0$ and elliptic for $0 < K < 4$.

(Boundary cases: $K = \pm 4 \Rightarrow \text{Trace}(J) = -2$, is unstable, can be solved numerically. $K = 0 \Rightarrow \text{Trace}(J) = 2$, is unstable as well.)
Problem #2

Simple pendulum under gravity:

\[ \mathcal{H} = \frac{1}{2} \dot{\phi}^2 - \cos(\phi) \]

**Part a.** Expand \( \cos(\phi) \) in Taylor series.

\[ \cos(\phi) = 1 - \frac{1}{2!}\phi^2 + \frac{1}{4!}\phi^4 - \frac{1}{6!}\phi^6 + \ldots \]

\[ \mathcal{H} = \frac{1}{2} \dot{\phi}^2 + \frac{1}{2} F \phi^2 - \frac{F}{8!} \phi^4 + \frac{F}{720} \phi^6 + \ldots \]

\{having dropped the constant term which will not affect dynamical equations.\}

Thus, \( C_1 = \frac{1}{2} F \), \( C_2 = \frac{1}{2} \), \( C_3 = -\frac{F}{8!} \), \( C_4 = \frac{F}{720} \).

Keep only quadratic term for now:

\[ \mathcal{H} = \frac{1}{2} \dot{\phi}^2 + \frac{1}{2} F \phi^2 - \frac{F}{8!} \phi^4 + o(\phi^6) \]

**Part b.** Using the generating function \( S_0 = \sin^2 \theta \) for the unperturbed H.O.

(i) \( \mathcal{J} = \frac{\partial S_0}{\partial \dot{\theta}} = \frac{1}{2} \sqrt{\frac{F}{\phi}} \cos^2 \theta \)

(ii) \( \mathcal{P} = \frac{\partial S_0}{\partial \theta} = \sqrt{\frac{F}{\phi}} \sin \theta \)

\[ \text{Eq.} \]
Thus, \( \cot \theta = \frac{\rho}{R^2} \)

\( R = (\frac{F}{G})^{1/2} \)

\( \csc^2 \phi = \frac{\rho^2 + R^2}{R^2} \)

and \( J = \frac{1}{2} R \rho^2 \left( \rho^2 + R^2 \right) \)

\( = \frac{1}{2} \frac{R^4}{R} \frac{1}{R^2} = \frac{R^2}{\sqrt{FG}} \)

or \( \lambda = \sqrt{FG} - J \) (as expected.)

from (i) and (ii), we have:

\( \rho^2 = 2RJ \cos^2 \omega \Rightarrow \frac{1}{6} \rho^2 = \frac{1}{2} \sqrt{FG} \left( \frac{23}{15} \right) \cos^2 \omega \)

\( \phi^2 = 2J \frac{1}{R} \sin^2 \omega \Rightarrow \frac{1}{4} \frac{2J^3}{R} \sin^2 \omega + \frac{1}{30} \frac{2J^3}{R^2} \sin^4 \omega + \cdots \)

\( \lambda = \omega J - \frac{1}{6} GJ^2 \sin^2 \omega + \frac{1}{30} \frac{GJ^3}{R^2} \sin^4 \omega + \cdots \)

\( \omega_0 = \sqrt{FG} \). Expanding in powers of \( \sin \omega \):

\( \lambda_0 = \omega_0 J \)

\( \lambda_1 = -\frac{2J^2}{48} \left( 3 - 4 \cos \omega + \cos 4\omega \right) \)

\( \lambda_2 = \frac{2J^3}{288\omega_0} \left( 10 - 15 \cos \omega + 6 \cos 4\omega - \cos 6\omega \right) \)
\[ \bar{\mathbf{r}} = \mathbf{r}_0 (\mathbf{j}) + \mathbf{r}_1 (\mathbf{j}, \bar{\mathbf{o}}) + \mathbf{r}_2 (\mathbf{j}, \bar{\mathbf{o}}) + \ldots \]

\[ \langle \mathbf{r}_0 (\mathbf{j}) \rangle = \omega_0 \mathbf{j} \]

\[ \langle \mathbf{r}_1 (\mathbf{j}, \bar{\mathbf{o}}) \rangle = -\frac{G \bar{\mathbf{j}}^2}{16} \]

\[ \langle \mathbf{r}_2 (\mathbf{j}, \bar{\mathbf{o}}) \rangle = \frac{G^2 \bar{\mathbf{j}}^3}{2 \pi \omega_0} \]

Thus:

\[ \omega = \frac{\mathbf{d} \mathbf{r}_0}{\mathbf{d} \mathbf{j}} = \omega_0 - \frac{1}{8} G \bar{\mathbf{j}} + \frac{G^2 \bar{\mathbf{j}}^2}{96 \omega_0} \]

As \( \omega \) has small machine accuracy to first order (\( \zeta > 0 \)).

\[ \text{Part 2.} \quad \frac{d S_1}{d \bar{\mathbf{o}}} = \mathbf{j}_1 - \langle \mathbf{j}_1 \rangle = \frac{G \bar{\mathbf{j}}^2}{4 \omega_0} \left( 4 \cos 2\bar{\mathbf{o}} - \cos 4\bar{\mathbf{o}} \right) \]

\[ \Rightarrow \frac{d S_1}{d \bar{\mathbf{o}}} = \frac{G \bar{\mathbf{j}}^2}{4 \omega_0} \left( 4 \cos 2\bar{\mathbf{o}} - \cos 4\bar{\mathbf{o}} \right), \text{ lowest order.} \]

\[ S_1 = \frac{G \bar{\mathbf{j}}^2}{4 \omega_0} \left( 2 \sin 2\bar{\mathbf{o}} - \frac{1}{4} \sin 4\bar{\mathbf{o}} \right) \]

\[ \text{Or.} \quad S_1 = \frac{G \bar{\mathbf{j}}^2}{192 \omega_0} \left( 8 \sin 2\bar{\mathbf{o}} - \sin 4\bar{\mathbf{o}} \right) \]
\[ \mathbf{Q} = \mathbf{0} - \frac{\partial}{\partial \mathbf{j}} \left( \mathbf{F}(\mathbf{j}, \mathbf{0}) \right) \]

\[ \mathbf{Q} = \mathbf{0} - \frac{6 \mathbf{j}}{16 \omega_0} \left( 8 \sin \theta - \sin \phi \mathbf{e}_\phi \right) \]

\[ \mathbf{J} = \mathbf{j} + \frac{\partial \mathbf{F}_1(\mathbf{j}, \mathbf{0})}{\partial \mathbf{0}} \]

\[ \mathbf{J} = \mathbf{j} + \frac{6 \mathbf{j}}{4 \mathbf{1} \omega_0} \left( 4 \cos \theta - \sin \phi \mathbf{e}_\phi \right) \]

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**Post 2.** Second-order corrections were carried out as shown above (see pg. 4 for \( \mathbf{h}_2 \), 5 for \( \mathbf{k}_2 \) and \( \mathbf{l}_2 \)).

Plotting \( \mathbf{w} \) as a function of \( \omega \mathbf{J} = \mathbf{E}_0 \),

\[ \mathbf{w} - \mathbf{w}_0 = \mathbf{w}_1 + \mathbf{w}_2 \]
part 6.

\[ f(x, y, t^1) = \sin kx \sin \pi y + \varepsilon \cos \pi x \cos \pi y. \]

\[ e = 0 : \quad f(x, y, t^1) = \sin kx \sin \pi y \]

(i) \[ \frac{\partial f}{\partial y} = u_x \quad , \quad (ii) \quad \frac{\partial f}{\partial x} = u_y \]

(i) \[ u_x = -k \sin kx \cos \pi y = \frac{\partial x}{\partial t} \]

(ii) \[ u_y = k \cos kx \sin \pi y = \frac{\partial y}{\partial t} \]

part c. Numerically integrating with MATLAB produces trajectories which remain close to a streamline for a while, then diverge to neighboring streamlines.
Problem 4

Shilnikov Phenomenon.

\[ \frac{dx}{dt} = f(x, y; \mu), \quad \frac{dy}{dt} = g(x, y; \mu) \]

\[ \begin{align*}
\pi_0 & \quad \mu < 0 \\
\pi_1 & \quad \mu = 0 \\
\pi_2 & \quad \mu > 0
\end{align*} \]

Part a: For a local map we may linearize the system above, e.g.

\[ \begin{align*}
\frac{dx}{dt} & = f_x x + f_y y \\
\frac{dy}{dt} & = g_x x + g_y y
\end{align*} \]

Thus

\[ \begin{pmatrix} x \\ y \end{pmatrix} = c_1 e^{\lambda_1 t} \mathbf{V}_1 + c_2 e^{\lambda_2 t} \mathbf{V}_2 \]

where \( \lambda_{1,2} \) are eigenvalues of Jacobian at the origin, \( \mathbf{V}_{1,2} \) are eigenvectors and \( c_{1,2} \) are specified by initial perturbation.
new, \( x(t) \approx x_0 e^{\lambda t} \) for small \( x_0 \)

and \( T \) is given by \( x(T) = x_0 e^{\lambda T} = \epsilon \)

\[
T = \frac{\ln \left( \frac{\epsilon}{x_0} \right)}{\lambda}
\]

where \( T \) is the time of flight between \( T_1 \) and \( T_2 \) in linearized region around the origin.

Furthermore, at \( T_1, \ y(T_1) = \epsilon e^{\gamma T} = \epsilon e^{\gamma \ln \left( \frac{\epsilon}{x_0} \right)} \)

or \( y(T_1) = \epsilon \cdot \left( \frac{\epsilon}{x_0} \right)^{\gamma/\lambda} \)

Finally, the map is written

\[
M(x, y; \tau) = \left( \sum \epsilon \cdot \left( \frac{\epsilon}{x} \right)^{\gamma/\lambda} \right)
\]

part b. Assuming, as discussed in lecture, that

the action of non-linearities amounts to taking

points on \( \Pi_1 \), distorting them by stretching and translating to \( \Pi_0 \) by sending \( y \to -x \), the

map should be:

\[
M_1 \equiv \left( \begin{array}{c} x' \\ y' \\ \end{array} \right) = \left( \begin{array}{cc} 0 & \epsilon \\ 0 & \epsilon \end{array} \right) \left( \begin{array}{c} x \\ y \\ \end{array} \right) + \left( \begin{array}{c} 0 \\ -1 \\ \end{array} \right)
\]

send \( y \to -x \) on \( \Pi_0 \)
We need only consider the equation for $x$. Thus,
\[ M_2(1 + x^2) = \]
\[ = M_2 \left( \varepsilon \left( \frac{\varepsilon}{x} \right)^{\frac{\varepsilon}{\mu}} \right) \]
\[ = \left( \begin{array}{c}
\varepsilon \\
\varepsilon^{1/2} x^{1/2}
\end{array} \right) \left( \begin{array}{c}
\frac{\varepsilon}{x} \\
\mu
\end{array} \right) + \left( \begin{array}{c}
\varepsilon \\
-\mu
\end{array} \right) \]
\[ \Rightarrow \lambda_{1,2} = c \cdot \varepsilon \left( \frac{\varepsilon}{x} \right)^{\frac{\varepsilon}{\mu}} - \mu. \]

Fixed points \[ x_{\lambda_{1,2}} = x_c = x^c \]
\[ x^c = a \varepsilon \left( \frac{\varepsilon}{x^c} \right)^{\frac{\varepsilon}{\mu}} - \mu \]
\[ = a \varepsilon \left( \frac{x^c}{\varepsilon} \right)^{\frac{\varepsilon}{\mu}} - \mu. \]

Stability is determined by \( \frac{dx_{\lambda_{1,2}}}{dx_c} \) as usual:
\[ \frac{dx_{\lambda_{1,2}}}{dx_c} = 0 \cdot \frac{dy_{\lambda_{1,2}}}{dx_c} \left( \frac{dy_{\lambda_{1,2}}}{dx_c} \right)^{-1} \]
\[ = 0 \cdot \frac{\mu x^c - \mu}{\lambda} \cdot \frac{x^c}{\mu} \]
\[ = \frac{\mu}{\lambda} \cdot \frac{x^c - \mu}{x^c}. \]
Thus, for \( \mu = 0 \), the fixed point is stable for \( \frac{|x|}{\lambda} < 1 \) and unstable for \( \frac{|x|}{\lambda} > 1 \), implying that we have a limit cycle for

\[
\frac{|x|}{\lambda} < 1.
\]