APM203 Problem Set #11
Solutions

Problem #1

Standard Map: $\theta_{n+1} = \theta_n + p_n \mod 2\pi$
$p_{n+1} = p_n + K \sin \theta_{n+1}$

To find fixed points (θ, p) solve equations

(i) $\theta = \theta + p \mod 2\pi$
(ii) $p = p + K \sin \theta$ $\quad (-\pi < p < \pi)$

(i) is satisfied by $p = 0$ in the interval $|p| < \pi$.

for $K = 0$, (ii) is trivially satisfied for any θ
if $K > 0$, (ii) $\Rightarrow$ $\theta = 0$ or π.

Thus, fixed points are:

$$\left\{ \begin{array}{lll} K = 0 : & (\theta, p) = (\theta, 0) & \text{any } \theta. \\ K > 0 : & (\theta, p) = (0, 0) \text{ or } (\pi, 0) & \end{array} \right\}$$

To compute the stability of these f.p.'s we turn to the Jacobian:

$$J = \begin{pmatrix} \frac{\partial \theta_{n+1}}{\partial \theta_n} & \frac{\partial \theta_{n+1}}{\partial p_n} \\ \frac{\partial p_{n+1}}{\partial \theta_n} & \frac{\partial p_{n+1}}{\partial p_n} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ K\cos(\underset{\uparrow \atop \theta_{n+1}}{\theta + p}) & 1 + K\cos(\underset{\uparrow \atop \theta_{n+1}}{\theta + p}) \end{pmatrix}$$

$$\text{Trace}(J) = 2 + K\cos(\theta + p)$$

Investigate fixed points for $K \neq 0$:

$(0,0)$: $\text{Trace}(J) = 2 + K$

Thus, $(0,0)$ is hyperbolic for $K > 0$, hyperbolic with reflection for $K < -4$ and elliptic for $-4 < K < 0$.

$(\pi, 0)$: $\text{Trace}(J) = 2 - K$

Similarly, $(\pi, 0)$ is hyperbolic for $K < 0$, hyperbolic with reflection for $K > 4$ and elliptic for $0 < K < 4$

(Borderline cases : $K = \pm 4 \Rightarrow \text{Trace}(J) = -2$, is unstable, can be shown numerically.

$K = 0 \Rightarrow \text{Trace}(J) = 2$, is <u>unstable</u> as well.)

Problem #2

Simple pendulum under gravity:

$$\mathcal{H} = \frac{1}{2} G p^2 - F \cos(\phi)$$

<u>part a.</u> Expand in Taylor series:

using $\cos(\phi) = 1 - \frac{1}{2!}\phi^2 + \frac{1}{4!}\phi^4 - \frac{1}{6!}\phi^6 + \dots$

$$\mathcal{H} = \underbrace{\frac{1}{2} G p^2 + \frac{1}{2} F \phi^2}_{\text{S.H.O. part.}} - \frac{F}{24}\phi^4 + \frac{F}{720}\phi^6 - \dots$$

{ having dropped the constant term which will not affect dynamical equations. }

thus $C_1 = \frac{1}{2} G$, $C_2 = \frac{1}{2} F$, $C = -\frac{F}{24}$, $\varepsilon = \frac{F}{}$

Keep only quadratic term for now:

$$\mathcal{H} = \frac{1}{2} G p^2 + \frac{1}{2} F \phi^2 - \frac{F}{24}\phi^4 + O(\phi^6).$$

<u>part b.</u> using the generating function $S_0 = \frac{1}{2}\sqrt{F/G}\, q^2 \cot\phi$ for the unperturbed H.O.

(i) $J = -\frac{\partial S_0}{\partial \phi} = \frac{1}{2}\sqrt{\frac{F}{G}}\,\phi^2 \csc^2\theta$

(ii) $p = \frac{\partial S_0}{\partial q} = \sqrt{\frac{F}{G}}\,\phi \cot\theta$

Thus, $\cot\theta = \dfrac{p}{R\phi}$ $\qquad R \equiv (F/G)^{1/2}$

$\Rightarrow \quad \csc^2\phi = \dfrac{p^2 + R^2\phi^2}{R^2\phi^2}$

and $\quad J = \frac{1}{2} R \cdot \frac{1}{R^2} \cdot (p^2 + R^2\phi^2)$

$= \frac{1}{2R} p^2 + \frac{1}{2} R\phi^2 = \dfrac{\mathcal{H}_0}{\sqrt{FG}}$

or $\quad \mathcal{H}_0 = \sqrt{FG}\, J \quad$ (as expected.)

from (i) and (ii), we have:

$p^2 = 2RJ\cos^2\theta \Rightarrow \frac{1}{2} G p^2 = \frac{1}{2}\sqrt{FG}\,(2J)\cos^2\theta$

$\phi^2 = \frac{2J}{R}\cdot\sin^2\theta \Rightarrow \frac{1}{2} F\phi^2 = \frac{1}{2}\sqrt{FG}\cdot(2J)\sin^2\theta$

Part c

$\mathcal{H} = \omega_0 J - \frac{1}{6}\cdot G J^2 \sin^4\theta + \frac{1}{90}\cdot\frac{G^2 J^3}{\omega_0}\sin^6\theta + \dots$

$\omega_0 \equiv \sqrt{FG}$. Expanding in powers of $\sin\theta$:

$\mathcal{H}_0 = \omega_0 J$

$\mathcal{H}_1 = -\frac{GJ^2}{48}(3 - 4\cos 2\theta + \cos 4\theta)$

$\mathcal{H}_2 = \frac{G^2 J^3}{2880\,\omega_0}(10 - 15\cos 2\theta + 6\cos 4\theta - \cos 6\theta)$

part d.

$$\bar{\mathcal{H}} = \mathcal{H}_0(\bar J) + \langle \mathcal{H}_1(\bar J, \bar\theta)\rangle + \langle \mathcal{H}_2(\bar J, \bar\theta)\rangle + \dots$$

$$\langle \mathcal{H}_0(\bar J)\rangle = \omega_0 \bar J$$

$$\langle \mathcal{H}_1(\bar J, \bar\theta)\rangle = -\frac{G\bar J^2}{16}$$

$$\langle \mathcal{H}_2(\bar J, \bar\theta)\rangle = \frac{G^2 \bar J^3}{288\,\omega_0}$$

thus: $$\omega = \frac{\partial \bar{\mathcal{H}}}{\partial \bar J} = \omega_0 \underbrace{- \frac{1}{8} G \bar J}_{\text{first order}} + \underbrace{\frac{G^2 \bar J^2}{96\,\omega_0}}_{\text{second order.}}$$

action increases with amplitude, thus ω has small negative correction to first order ($G>0$.)

part e.

$$\omega \frac{\partial S_1}{\partial \bar\theta} = \mathcal{H}_1 - \langle \mathcal{H}_1\rangle = \frac{G\bar J^2}{48}\left(4\cos 2\bar\theta - \cos 4\bar\theta\right)$$

$$\Rightarrow \frac{\partial S_1}{\partial \bar\theta} = \frac{G\bar J^2}{48\,\omega_0}\left(4\cos 2\bar\theta - \cos 4\bar\theta\right),$$ * lowest order.

$$S_1 = \frac{G\bar J^2}{48\,\omega_0}\left(2\sin 2\bar\theta - \frac{1}{4}\sin 4\bar\theta\right)$$

$$\left\{ \text{or. } S_1 = \frac{G\bar J^2}{192\,\omega_0}\left(8\sin 2\bar\theta - \sin 4\bar\theta\right) \right\}$$

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part f.

$$\theta = \bar{\theta} - \frac{\partial S_1(\bar{J},\bar{\theta})}{\partial \bar{J}}$$

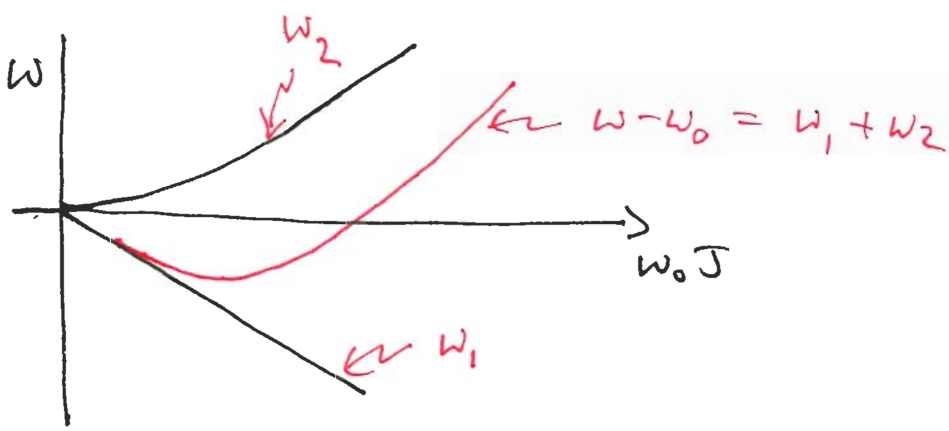
$$= \bar{\theta} - \frac{2G\bar{J}}{192\,\omega_0}\left(8\sin 2\bar{\theta} - \sin 4\bar{\theta}\right)$$

$$\boxed{\theta = \bar{\theta} - \frac{G\bar{J}}{96\,\omega_0}\left(8\sin 2\bar{\theta} - \sin 4\bar{\theta}\right)}$$

$$J = \bar{J} + \frac{\partial S_1(\bar{J},\bar{\theta})}{\partial \bar{\theta}}$$

$$\boxed{J = \bar{J} + \frac{G\bar{J}}{48\,\omega_0}\left(4\cos\bar{\theta} - \sin\bar{\theta}\right)}$$

part g. Second-order corrections were computed above (see pg. 4 for H_2, 5 for $\langle H_2\rangle$ and ω.) Plotting ω as a function of $\omega_0 J = E_0$,



Problem #3

part b. $\psi(x,y,t) = \sin kx \sin \pi y + \varepsilon \cos \omega t \times \cos kx \cos \pi y.$

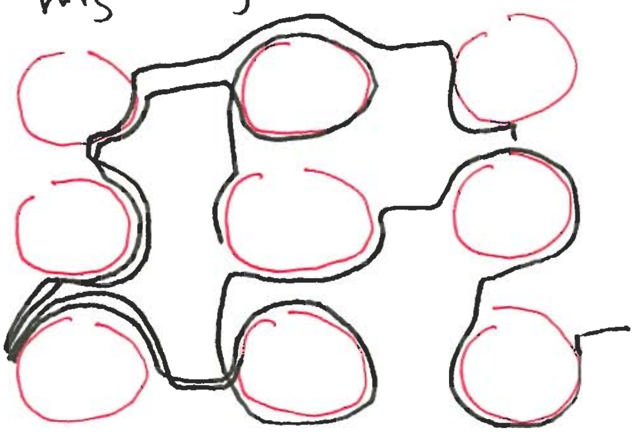
$\varepsilon = 0: \quad \psi(x,y,t) = \sin kx \sin \pi y$

(i) $-\dfrac{\partial \psi}{\partial y} = v_x$, (ii) $\dfrac{\partial \psi}{\partial x} = v_y$

(i) $\Rightarrow \quad v_x = -\pi \sin kx \cos \pi y = \dfrac{dx}{dt}$

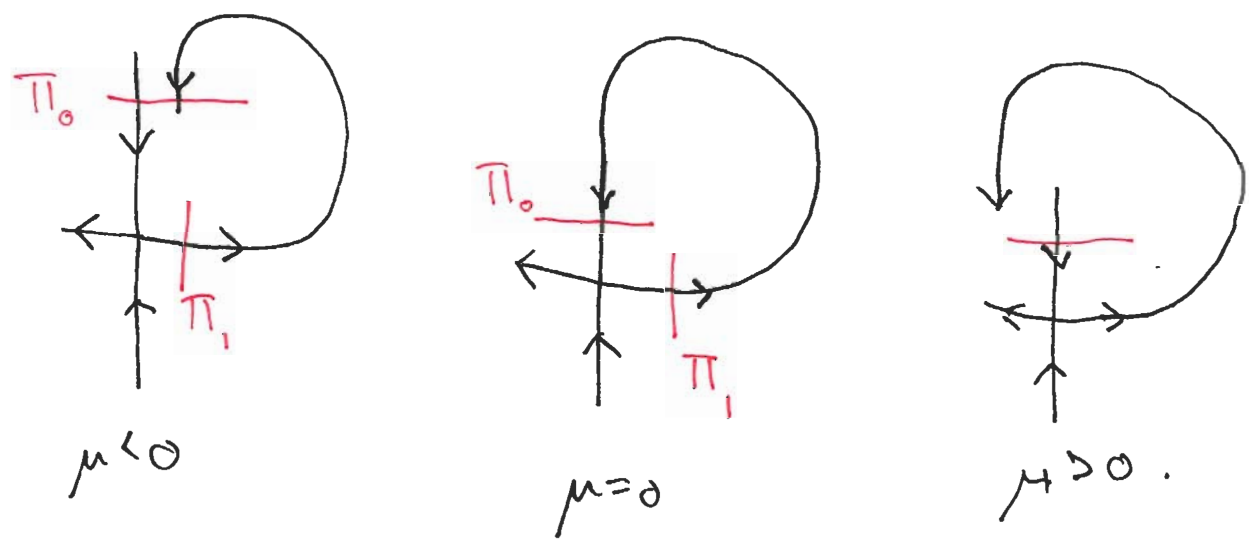
(ii) $\Rightarrow \quad v_y = k \cos kx \sin \pi y = \dfrac{dy}{dt}$

part e. Numerically integrating with Matlab produces trajectories which remain close to a streamline with $\varepsilon = 0$ (part b.) for a while, then diverge to neighboring streamlines



Problem # 4 Shilnikov Phenomenon.

$$\frac{dx}{dt} = f(x, y; \mu) \quad , \quad \frac{dy}{dt} = g(x, y; \mu)$$



part a. For a local map we may linearize the system above. e.g.

$$\frac{dx}{dt} = f_x x + f_y y$$

$$\frac{dy}{dt} = g_x x + g_y y$$

thus $\begin{pmatrix} x \\ y \end{pmatrix} = c_1 e^{\lambda_1 t} \vec{v}_1 + c_2 e^{\lambda_2 t} \vec{v}_2$

where $\lambda_{1,2}$ are eigenvalues of Jacobian at the origin, $\vec{v}_{1,2}$ are eigenvectors and $c_{1,2}$ are specified by initial perturbation.

Now, $x(t) \cong x_0 \, e^{\lambda t}$ for small x_0

and T is given by $x(T) = x_0 \, e^{\lambda T} = \varepsilon$

$$\Rightarrow \quad T = \frac{1}{\lambda} \log\left(\frac{\varepsilon}{x_0}\right)$$

where T is the time of flight between Π_0 and Π_1 in linearization region around the origin.

Furthermore, at T, $y(T) = \varepsilon \, e^{\gamma T} = \varepsilon \, e^{\frac{\gamma}{\lambda} \log\left(\frac{\varepsilon}{x_0}\right)}$

or $y(T) = \varepsilon \cdot \left(\frac{\varepsilon}{x_0}\right)^{\gamma/\lambda}$

Finally, the map is written

$$\boxed{M(x, y=\varepsilon) = \left(\varepsilon, \; \varepsilon\left(\frac{\varepsilon}{x}\right)^{\gamma/\lambda}\right)}$$

<u>part b.</u> Assuming, as discussed in lecture, that the action of nonlinearities amounts to taking points on Π_1 distorting them by stretching and translating to Π_0 by sending $y \to x$, the map should be:

stretch on Π_1 then send $y \to x$.

shift in x-direction by $-\mu$ as illustrated

$$M_2 \equiv \begin{pmatrix} x' \\ \varepsilon \end{pmatrix} = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \varepsilon \\ y \end{pmatrix} + \begin{pmatrix} -\mu \\ \varepsilon \end{pmatrix}$$

send $y \to \varepsilon$ on Π_0

part c. We need only consider the equations for x. Thus, $M_2(M_1(\vec{x})) =$

$$= M_2\left(\varepsilon,\ \varepsilon\cdot\left(\frac{\varepsilon}{x}\right)^{\gamma/\lambda}\right)$$

$$= \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \varepsilon \\ \varepsilon\left(\frac{\varepsilon}{x}\right)^{\gamma/\lambda} \end{pmatrix} + \begin{pmatrix} -\mu \\ \varepsilon \end{pmatrix}$$

$$\Rightarrow \boxed{x_{n+1} = a\varepsilon\cdot\left(\frac{\varepsilon}{x_n}\right)^{\gamma/\lambda} - \mu.}$$

part d. Fixed points $\Rightarrow$ $x_{n+1} = x_n \equiv x^*$

$$x^* = a\varepsilon\cdot\left(\frac{\varepsilon}{x^*}\right)^{\gamma/\lambda} - \mu$$

$$= a\varepsilon\cdot\left(\frac{x^*}{\varepsilon}\right)^{|\gamma|/\lambda} - \mu.$$

Stability is determined by $\frac{dx_{n+1}}{dx_n}$ as usual.

$$\left.\frac{dx_{n+1}}{dx_n}\right|_{x^*} = a\cdot\frac{|\gamma|}{\lambda}\left(\frac{x^*}{\varepsilon}\right)^{|\gamma|/\lambda}\frac{\varepsilon}{x^*}$$

$$= \cancel{a}\cdot\frac{|\gamma|}{\lambda}\cdot\frac{x^*+\mu}{\cancel{a}\cancel{\varepsilon}}\cdot\frac{\cancel{\varepsilon}}{x^*}$$

$$= \frac{|\gamma|}{\lambda}\cdot\frac{x^*+\mu}{x^*}$$

Thus, for $\mu = 0$, the fixed point is stable for $\frac{|\gamma|}{\bar{\lambda}} < 1$ and unstable for $\frac{|\gamma|}{\bar{\lambda}} > 1$, implying that we have a limit cycle for $\frac{|\gamma|}{\bar{\lambda}} < 1$.