Homework #6
Nonlinear dynamics and chaos

1. Calculate a numerical approximation for
   \[
   \lim_{n \to \infty} \frac{r_n - r_{n-1}}{r_{n+1} - r_n}
   \]
   for the following two maps
   \[
   x_{n+1} = r \sin(\pi x_n), \quad 0 \leq r \leq 1 \quad \text{(the sine map)} \quad (1)
   
   x_{n+1} = r - x_n^4 \quad (2)
   \]
   by iterating the maps for different \(r\)-values and finding the \(r\) values at which period doubling(s) occur. Compare the results to \(\delta\) for the logistic map and explain.

2. (Strogatz 10.7.3,4) some simple renormalization-related issues:
   (a) Show that if \(g(x)\) is a fixed point of the doubling transformation, that is,
       \[
       g(x) = -\alpha g \left( \frac{x}{-\alpha} \right) \equiv T[g],
       \]
       so is \(\mu g(x/\mu)\).
   (b) show that \(g(x)\) crosses the line \(y = \pm x\) an infinite number of times by showing that
       if \(x^*\) is a fixed point of \(g(x)\), so is \(-\alpha x^*\).
   (c) Calculate an approximation to the universal \(\alpha\) for the period doubling route to
       chaos. Start with the map \(f(x, r) = r - x^2\), assume a two-term expansion for the
       universal function:
       \[
       g(x) = 1 + c_2 x^2.
       \]
       and calculate \(c_2\) and \(\alpha\) that approximately satisfy the functional equation for \(g(x)\).

3. Show that
   \[
   g_{i-1}(x) = (-\alpha) g_i \left( \frac{x}{\alpha} \right) \equiv T[g_i(x)]
   \]
   Explain each stage in your derivation. (Schuster derives this, so you just need to explain
   what he does).

4. Do only one of the following two questions:
   (a) read, understand, and reproduce the approach of Strogatz “renormalization for
       pedestrians” pages 384-387 in order to analytically calculate an approximate to
       both \(\delta\) and \(\alpha\) for a quadratic maximum map. Skip example 10.7.2, but do example
       10.7.3.
   (b) Challenge question/ extra(!) credit: First the easier part: Find \(\alpha\) for quartic func-
       tions (such as \(x_{n+1} = r - x_n^4\)) using the approach of question 2c
       Next: a challenge in the best sense of the word (i.e. I have not tried this myself, and
       I don’t know that it is possible). Follow the approach of Strogatz “renormalization
       for pedestrians” on page 384-387 in order to analytically calculate both \(\delta\) and \(\alpha\) for
       a quartic maximum function (such as \(x_{n+1} = r - x_n^4\)). Compare your analytically
       derived results to the numerical approximation for \(\delta\) from question 1.