Homework #4 Solutions

APM 203 – Prof Eli Tziperman

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Problem #1 From Strogatz, Problem 6.6.8 page 191.

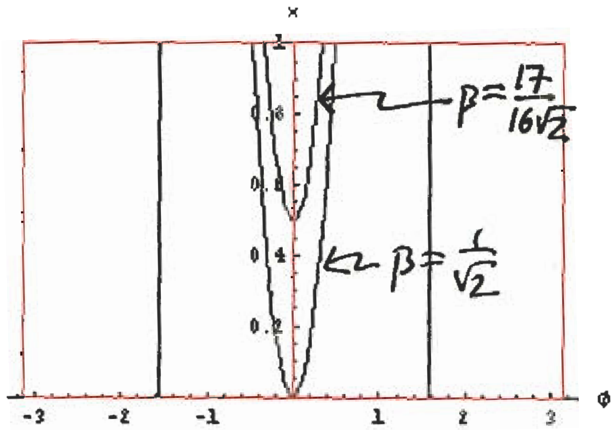
$$\dot{x} = \frac{\sqrt{2}}{4}\, x\,(x-1)\sin(\phi)$$
$$\dot{\phi} = \frac{1}{2}\left[\beta - \frac{1}{\sqrt{2}}\cos(\phi) - \frac{1}{8\sqrt{2}}\, x\cos(\phi)\right]$$

part a. Note $\dot{x}$ is odd in ϕ while $\dot{\phi}$ is even in ϕ. Thus, if we perform the transformation $x \to x$, $\phi \to -\phi$ and $t \to -t$, the system is unaltered $\Rightarrow$ REVERSIBLE. Incidentally, this means no trajectories may run along the $\phi = 0$ line.

part b. $\dot{x}$ nullclines are the sides of the domain $D = \{(\phi, x);\ \phi \in [-\pi, \pi],\ x \in [0, 1]\}$ and the line $\phi = 0$. The $\dot{\phi}$ nullcline is

$$x(\phi) = 8\left(\frac{\sqrt{2}\beta}{\cos\phi} - 1\right)$$

Plotted below are the $\dot{x}$ nullclines (red) and $\dot{\phi}$ nullclines for $\beta = \frac{1}{\sqrt{2}}$ and $\beta = \frac{17}{16\sqrt{2}}$ where the latter is halfway between $\beta = \frac{1}{\sqrt{2}}$ and $\beta = \frac{9}{8\sqrt{2}}$. The nullclines clearly intersect in three different places for $\frac{1}{\sqrt{2}} < \beta < \frac{9}{8\sqrt{2}}$.



The fixed points are immediately found to be

$(\phi^*, x^*) = \left(\pm \cos^{-1}\left(\frac{8\beta}{9\sqrt{2}}\right), 1\right)$ along the top edge and

$(\phi^*, x^*) = \left(0, 8(\sqrt{2}\beta - 1)\right)$ along the $\phi = 0$ line.

Compute the linearization (Jacobian) matrix of the last:

$$J = \begin{pmatrix} 0 & \frac{\sqrt{2}}{4} x^*(x^*-1)\cos\phi^* \\ -\frac{1}{82\sqrt{2}}\cos\phi^* & 0 \end{pmatrix} \Rightarrow \lambda_\pm = \pm\sqrt{x^*(x^*+1)}/8$$

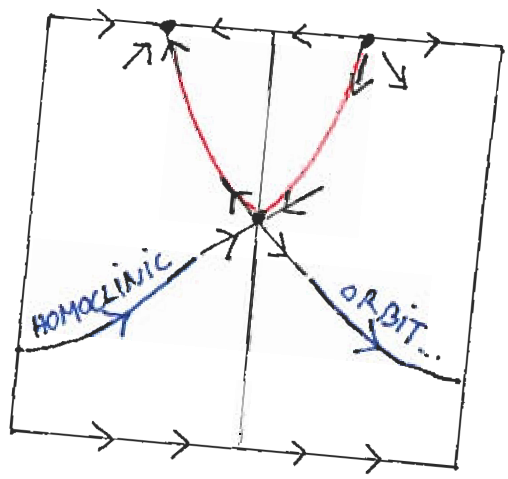
but $0 \le x^* \le 1$

$\Rightarrow \lambda_\pm \gtrless 0$ i.e. <u>SADDLE</u>

By simply investigating the behavior of $\dot{x}$ and $\dot{\phi}$ around the $x^* = 1$ f.p.'s one can deduce that the $\phi^* > 0$ f.p. is unstable while the $\phi^* < 0$ f.p. is stable (take small $\Delta\phi$ and Δx independently). $\boxed{2}$

Notice, furthermore, that at $x=0$, $\dot{\phi}>0$ for all β under consideration. Next, the only arrangements of stable (unstable) manifolds of the saddle are:

BUT, notice that $\dot{x}<0$ $\forall \phi>0$. Thus a. is the local behavior around the saddle. Plot the information gathered:



The $\dot{\phi}$ nullcline is plotted in red. Below it $\dot{\phi}>0$. For $\phi>0$, $\dot{x}<0$. Thus, the unstable manifold traveling into the $\phi>0$ plane moves down and to the right at all times. It cannot meet the $x=0$ solution by uniqueness thus it wraps around the cylinder like a belt:

We also use information that $\dot{x}=0$ on $\phi=\pm\pi$ to infer slope.

→ Same argument applies to any trajectory beginning at $\phi=0$ under saddle node ⇒ band of closed orbits.

part c. As $\beta \to \frac{1}{\sqrt{2}}$, $x^* \to 0$. The homoclinic orbit in blue above has $x(\phi) \leq x^*$ $\forall \phi$

⇒ $x(\phi)\to 0$ $\forall\phi$ thus, the homoclinic orbit becomes the $x=0$ orbit and chokes off closed orbits.

part d.

$\beta < \frac{1}{\sqrt{2}}$ new f.p.'s : $(\phi^*, x^*) = (\pm\cos^{-1}(\sqrt{2}\beta), 0)$

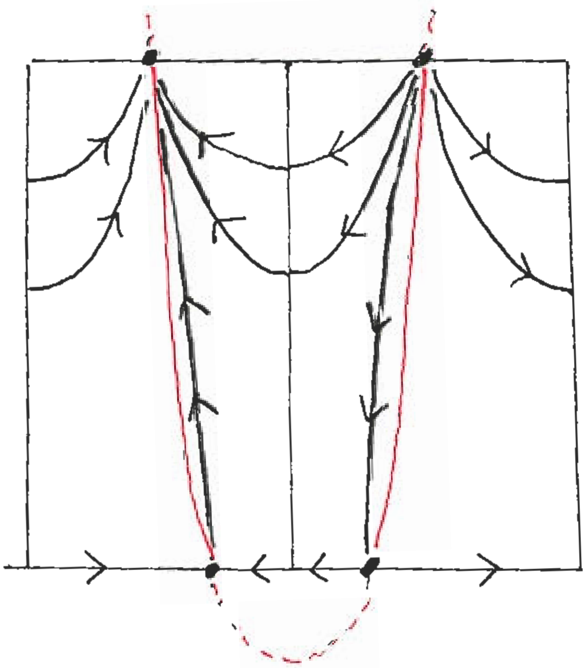
$$J = \begin{pmatrix} -\frac{\sqrt{2}}{4}\sin\phi^* & 0 \\ -\frac{1}{16\sqrt{2}}\cdot\sqrt{2}\,\beta & \frac{1}{2\sqrt{2}}\sin\phi^* \end{pmatrix}$$

Note $\mathrm{Trace}(J) = 0$, $\Delta(J) = -\frac{1}{8}\sin^2\phi^* < 0$

$\Rightarrow$ a saddle for both (ϕ^*, x^*) on $x = 0$.

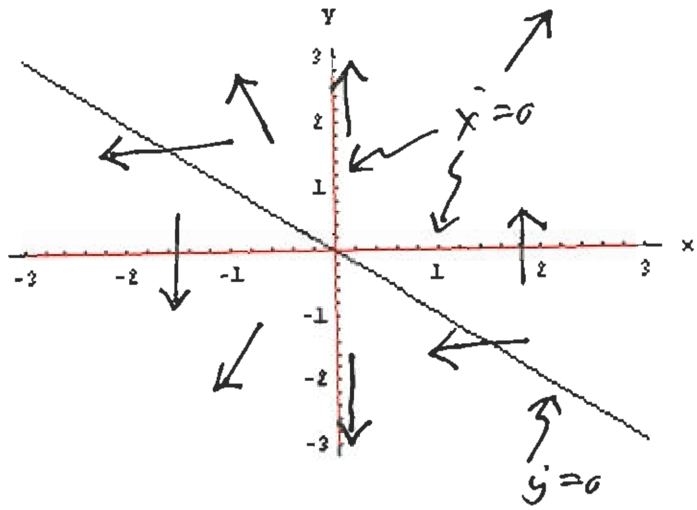
Nothing changes regarding our argument for the stability / instability of f.p.'s on $x = 1$ line. [redacted]

Flow looks thus:



Problem #2

$$\dot{x} = xy \; ; \quad \dot{y} = x + y$$



$\left\{ \begin{array}{c} \text{one fixed point} \\ \text{at } (0,0). \end{array} \right\}$

By observation, we can note that the arrows twist by a net angle $\Delta\phi_1 = 0$ in the 1^{st} quadrant, by $\Delta\phi_2 = \pi$ in the second quadrant, by $\Delta\phi_3 = 0$ in the 3^{rd}, and $\Delta\phi_4 = -\pi$ in the 4^{th}.

Thus $\frac{1}{2\pi}[\phi]_c = \sum_{i=1}^{4} \Delta\phi_i = 0$

Problem #3

a. $L(x,y) = x^2 + y^2$.

Then $$\frac{dL}{dt} = \frac{\partial L}{\partial x} \cdot \dot{x} + \frac{\partial L}{\partial y} \cdot \dot{y}$$
$$= 2x \cdot (y - x^3) + 2y \cdot (-x - y^3)$$
$$= -2(x^4 + y^4) \leq 0 \implies \text{NO LIMIT CYCLE.}$$

b. Let $\dot{\vec{x}} = \vec{f}(\vec{x})$ be a gradient system s.t.

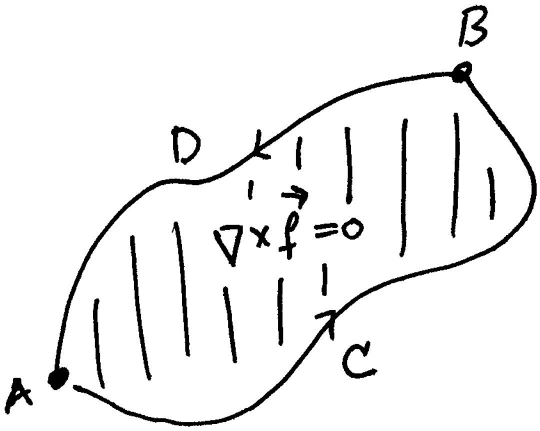
$\dot{\vec{x}} = -\nabla V$. Then $\vec{f}(\vec{x}) = \left(-\frac{\partial V}{\partial x}, -\frac{\partial V}{\partial y}\right)$.

Furthermore, if $\frac{\partial f}{\partial y} - \frac{\partial g}{\partial x} = 0$, we wish to demonstrate $\vec{f} = -\nabla V$. Note that the converse is easy to show

since $\vec{f} = -\nabla V \Rightarrow \nabla \times \vec{f} = \left(\frac{\partial f}{\partial y} - \frac{\partial g}{\partial x}\right)\hat{z} = \nabla \times (-\nabla V)$

$= 0$.

Assume we have $\nabla \times \vec{f} = 0$ [i.e. $\frac{\partial f}{\partial y} - \frac{\partial g}{\partial x} = 0$.]



Then, $\oint_{ACBDA} \vec{f} \cdot d\vec{r} = \int (\nabla \times \vec{f}) \cdot d\vec{\sigma}$
$= 0$

$\Rightarrow \int_{ACB} \vec{f} \cdot d\vec{r} = \int_{ADB} \vec{f} \cdot d\vec{r}$

i.e. line integral independent of path.

Thus $\int_A^B \vec{f} \cdot d\vec{r} = -V(B) + V(A)$

taking gradient of both sides and letting B vary: $B \equiv \vec{x}$
and A = origin $\rightarrow \boxed{-\nabla V = \vec{f}(\vec{x})}$

c. We have $\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(y + 2xy) = 1 + 2x$

$\frac{\partial g}{\partial x} = \frac{\partial}{\partial x}(x + x^2 - y^2) = 1 + 2x$

i.e. $\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x} \Rightarrow \vec{f} = -\nabla V \Rightarrow$ NO LIMIT CYCLE

OPTIONAL PROBLEMS

Problem #4

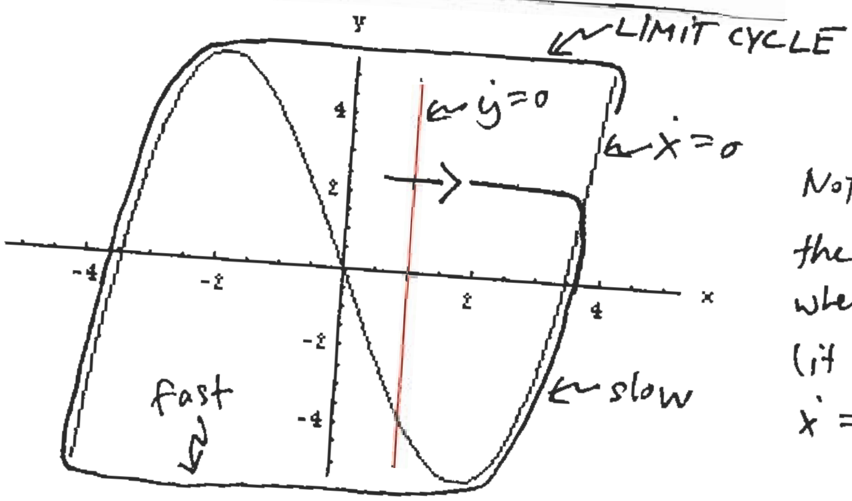
Estimate the period of the limit cycle of the system $\ddot{x} + k(x^2-4)\dot{x} + x = 1$ $\quad k \gg 1$.

Let $F(x) = \frac{1}{3}x^3 - 4x$

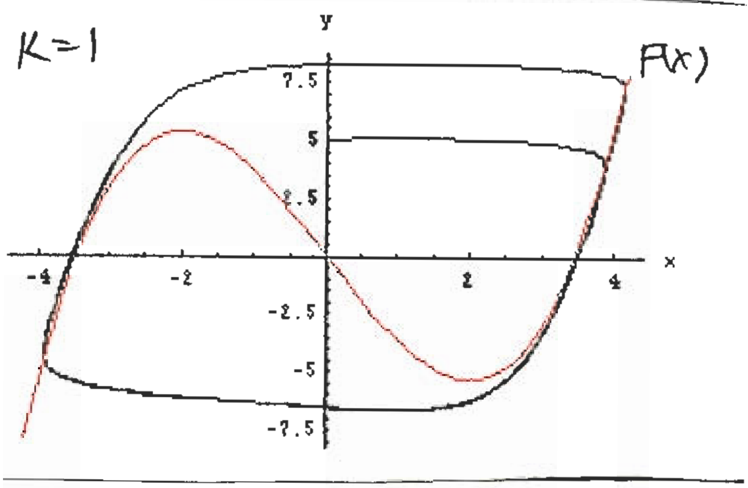
then $$\frac{d}{dt}\Big[\underbrace{\dot{x} + kF(x)}_{\equiv\, \omega}\Big] = \ddot{x} + k(x^2-4)\dot{x} = 1-x$$

$$\Rightarrow \quad \dot{\omega} = 1-x$$
$$\dot{x} = \omega - kF(x) = k\left(\frac{\omega}{k} - F(x)\right)$$

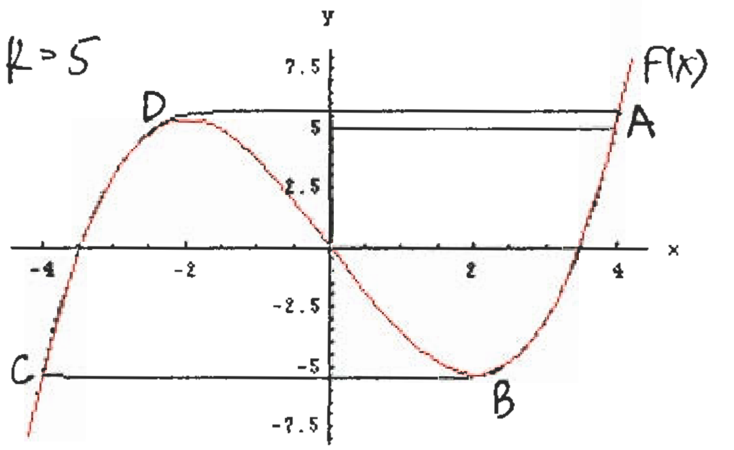
letting $y = \frac{\omega}{k}$, $\begin{cases} \dot{y} = \frac{1}{k}(1-x) \\ \dot{x} = k(y - F(x)) \end{cases}$



Note, the trajectory "hugs" the $\dot{x}$ nullcline $F(x)$, when it gets to the outside (it does so vertically, with $\dot{x} = 0$)



Numerical solutions to the system for K=1 and K=5. Note how $y = F(x)$ is a very good approximation along the slow branches for $K > 1$.



$$\text{Period} = T_{AB} + T_{CD}$$

Find x_B: this is the shoulder where $F'(x) = 0$ thus,

$$x_B^2 - 4 = 0 \Rightarrow x_B = 2$$

Furthermore, $x_D = -2$.

Also, $F(x_D) = F(x_A)$. Now

$$F(x_D) = \tfrac{1}{3}(-2)^3 - 4(-2) = -\tfrac{16}{3}$$

thus $\frac{1}{3}x_A^3 - 4x_A = -\frac{16}{3}$. $\Rightarrow$ $(x_A+4)(x_A-2)^2 = 0$

$x_A \neq -4 \Rightarrow x_A = 2$. Similarly, $x_C = -4$.

So: $$T_{AB} = \int_{x_A}^{x_B} \frac{dx}{\dot{x}}$$

we have $y = F(x)$ to an excellent approximation

$$\Rightarrow \dot{y} = F'(x)\dot{x} = (x^2-4)\dot{x}$$

or $$\dot{x} = \frac{(1-x)}{k(x^2-4)}$$

$$T_{AB} = k\int_{x_A}^{x_B} \frac{x^2-4}{1-x}\,dx$$

let $x = x'+1 \Rightarrow$ $$T_{AB} = k\int_2^4 \frac{(x'+1)^2-4}{x'}\,dx'$$

$$T_{AB} = k\int_2^4 x'\,dx' + k\int_2^4 2\,dx' - k\int_2^4 \frac{3}{x'}\,dx'$$

$$= -k\,(3\log 3 - 8)$$

and $T_{CD} = \left(4 - 3\log 5 + 3\log 3\right)K$

thus Period $\simeq K\left(8 - 3\log 3 + 4 - 3\log 5 + 3\log 3\right)$

$$= K\left(12 - 3\log 5\right).$$

Problem #5

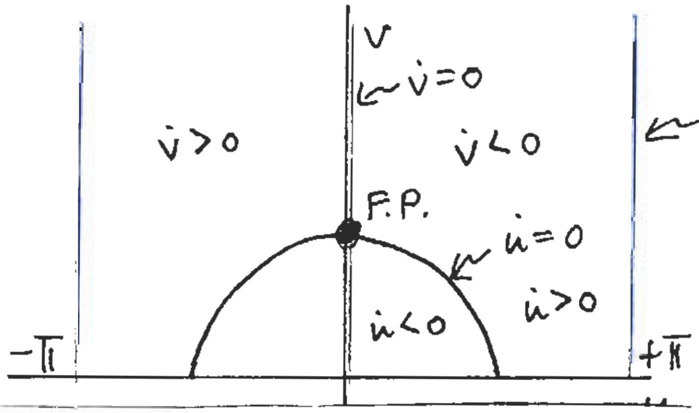
Glider problem.

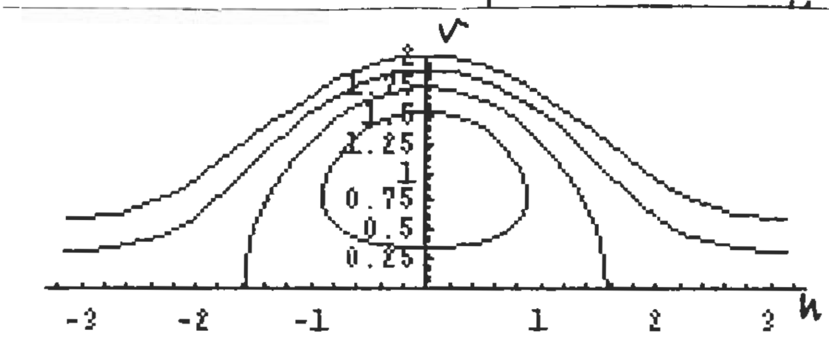
v = speed of glider

u = angle flight path makes with horizontal.

No drag:
$$\frac{dv}{dt} = -\sin u$$
$$v\,\frac{du}{dt} = -\cos u + v^2$$

Notice phase space is a half-infinite cylinder ($v \geq 0$) and the system is REVERSIBLE. Sketch phase space on a rectangle:





Analysis of the nullclines suggests robust (nonlinear) centers around F.P. and wavy right-moving solutions for v large enough.

Reversibility $\Rightarrow$ periodic sol'ns in 2π.

part b. "Find an exact expression"

$\dot{v}$ and $\dot{u}$ give $\dfrac{dv}{du} = \dfrac{-\sin u}{(-\cos u)/v + v}$

$\Rightarrow (-\cos u + v^2)\, dv = -v \sin u \, du$

or $(3v^2 - 3\cos u)\, dv + (3v \sin u)\, du = 0$

Note $\Rightarrow$ $f\, dv + g\, du = 0$ where $\dfrac{\partial g}{\partial v} = \dfrac{\partial f}{\partial u}$

thus $(f, g) = \nabla \Psi$. $\Psi = v^3 - 3v\cos u + \text{const.}$

$$\frac{\partial \Psi}{\partial v} dv + \frac{\partial \Psi}{\partial u} du \equiv d\Psi = 0 \Rightarrow \Psi = \text{constant.}$$

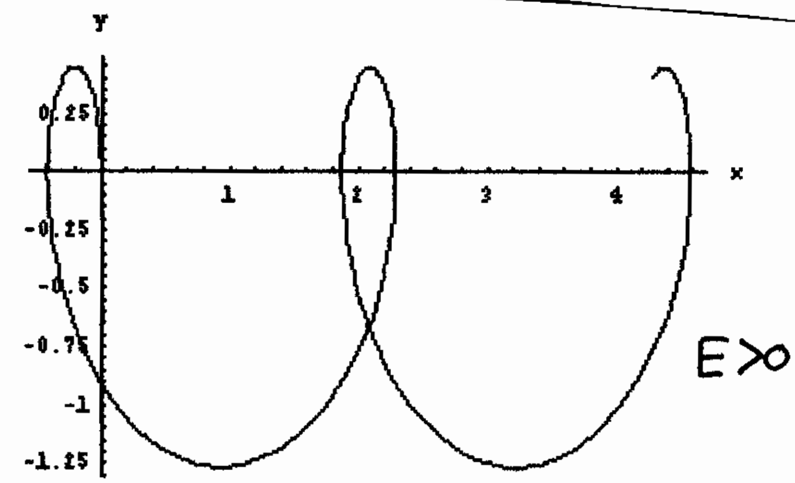
thus, $\boxed{v^3 - 3v\cos u = C}$

part c. The separatrix separates two regions of phase space with qualitatively different trajectories. The difference is that, under the separatrix, the system is confined to a range of angles within $(-\pi/2, \pi/2)$. outside of the separatrix, the system can take on any angle.

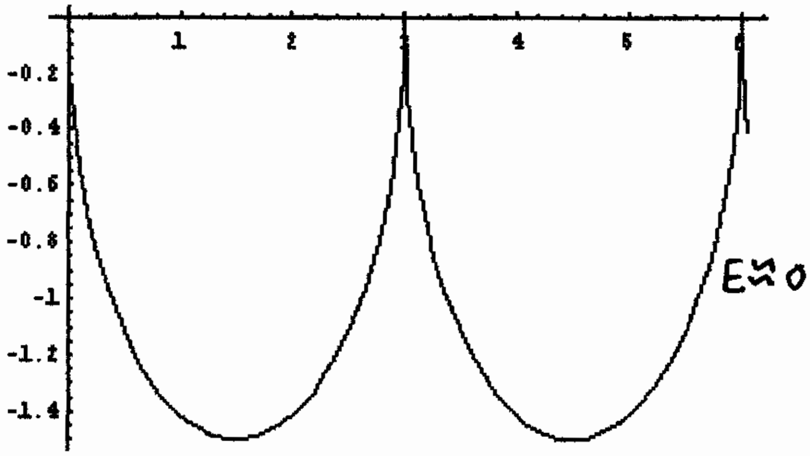
Let's use this information to find C of the separatrix.
We know that the only difference between a trajectory barely within the separatrix with a trajectory barely outside is that the latter has a solution $v \neq 0$ for all u outside of $(-\pi/2, \pi/2)$. Thus, the separatrix should have $v = 0$ as a root $\Rightarrow$ $\boxed{C = 0}$

Thus, $v = \sqrt{3\cos u}$ and $v = 0$ form the separatrix.

part d. Mathematica generates glider trajectories effortlessly:



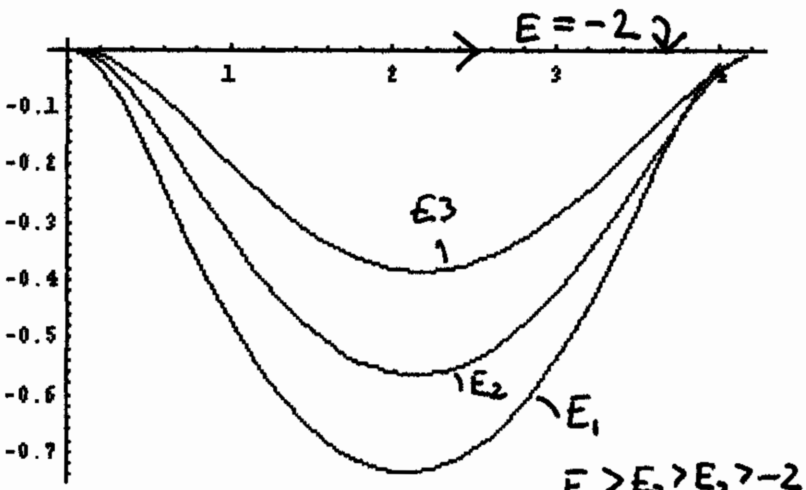
All angles explored. Glider loops as it travels forward. Spends relatively little time looping.



Glider flips from $\pi/2$ to $-\pi/2$ almost instantly at cusps. Now we go from

motion $(E>0)$ to

$(E<0)$ The glider "stalls."



Now $E<0$. The glider happily goes forth exploring a limited range of angles of inclination. At $E=-2$, the glider flies ahead at $v=1$, $u=0$ balanced by lift and gravity perfectly.

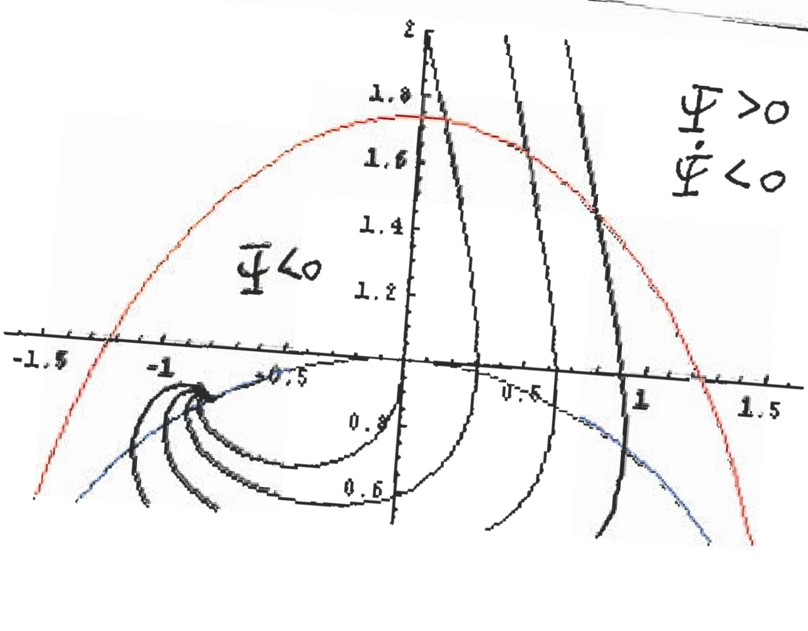
part e.

Now we put in drag:

$$\frac{dv}{dA} = -\sin u - Dv^2$$

$$v\frac{du}{dA} = -\cos u + v^2$$

Note the system is no longer time-reversible. The trajectories do not close on themselves. The fixed point is $(u^*, v^*) = \left(-\tan^{-1} D, (1+D^2)^{-1/4}\right)$. And has λ with negative real part indicating a stable spiral. Furthermore, the energy function $v^3 - 3v\cos u \equiv \Psi$ has the property $\dot{\Psi} = 0$ on $v = \cos u$ and $\dot{\Psi} < 0$ for all points such that $\Psi > 0$. Thus the glider must get trapped into the $E<0$ region. There it ultimately finds the F.P.



orbits come in from positive Ψ region and find a stable fixed point under old separatrix. Each orbit illustrated started with different E, same D. (D = 1.)