APM 203 Homework #3
SOLUTIONS

(a) Linearized 2d systems

for any system $\dot{\vec{x}} = A\vec{x}$, $\vec{x}^* = 0$ is a fixed point. It is the only fixed point if A is not singular. None of the following A's are singular, thus the origin is the unique f.p.

- $\dot{x} = x - y$ $\quad \dot{y} = x + y$

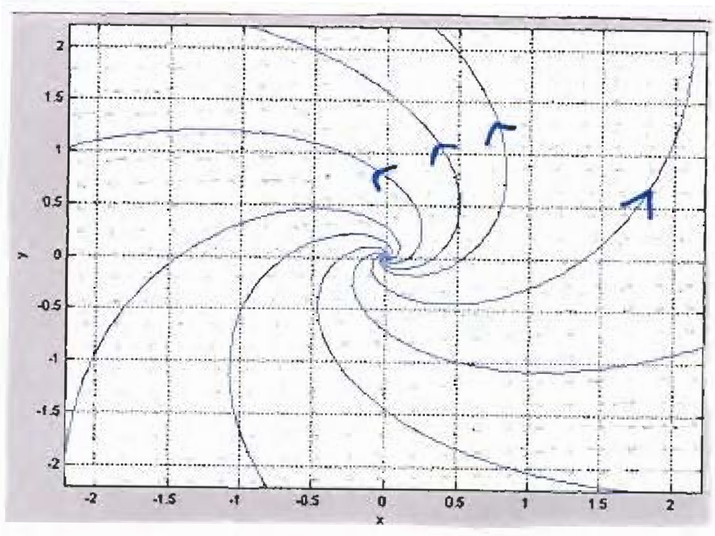
matrix form: $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & +1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

eigenvalues → $(1-\lambda)^2 + 1 = 0 \Rightarrow 1 - \lambda = \pm i$ or $\boxed{\lambda_\pm = 1 \pm i}$

eigenvectors → $\begin{pmatrix} \mp i & -1 \\ 1 & \mp i \end{pmatrix} \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} = 0 \Rightarrow \boxed{\begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} = \begin{pmatrix} \pm i \\ 1 \end{pmatrix}}$

The real part of the eigenvalue complex conjugate pair tells us the origin is UNSTABLE. The imaginary part gives oscillation ⇒ A SPIRAL.

A plot of the trajectories shows the spiral is <u>CCW</u>.



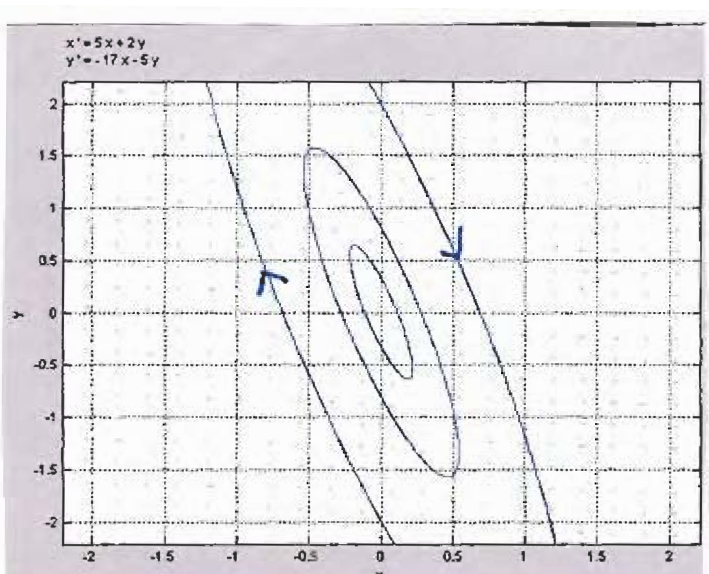
to generate these plots visit www.courses.harvard.edu/~apm147/assignments/Mattes_Files_PS5. (Thanks to Dorian for pointing this out to me.)

• $\dot{x} = 5x + 2y$, $\dot{y} = -17x - 5y$

matrix form : $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 5 & 2 \\ -17 & -5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

eigenvalues → $(5-\lambda)(-5-\lambda) + 34 = 0 \Rightarrow \boxed{\lambda_{\pm} = \pm 3i}$

Thus, the origin is a <u>CENTER</u>. There are no de-stabalizing nonlinear terms. The trajectories can be shown to be ellipses with semi-major axis tilted from the y-axis at angle $\theta = 16.845... = \frac{1}{2}\tan(2/3)$. (see Figure). Plot shows trajectories flow CW.



eigenvectors:

$\begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} = \begin{pmatrix} -5 \mp 3i \\ 17 \end{pmatrix}$

• $\dot{x} = 5x + 10y \qquad \dot{y} = -x - y$

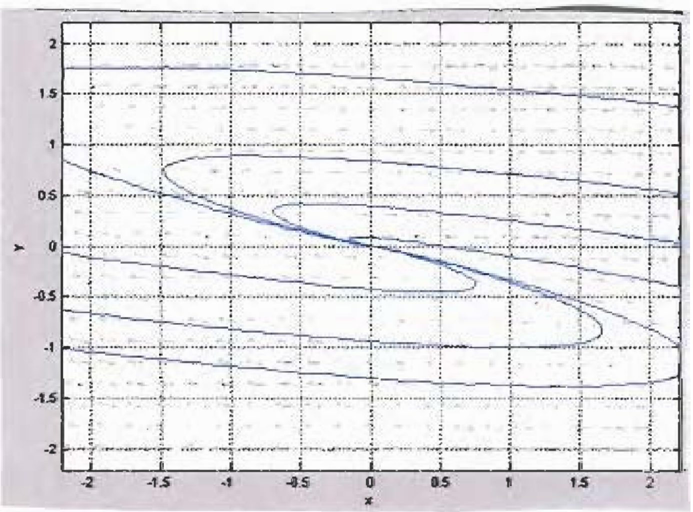
matrix form : $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 5 & 10 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

eigenvalues : $(5-\lambda)(-1-\lambda) + 10 = 0 \Rightarrow \lambda_{\pm} = 2 \pm i$

$Re(\lambda) > 0 \Rightarrow$ UNSTABLE, imaginary part $\Rightarrow$ spiral

eigenvectors : $\begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} = \begin{pmatrix} -3 \mp i \\ 1 \end{pmatrix}$

Plots show trajectories flowing CW.



(b) Nonlinear 2d SYSTEMS

• $\dot{x} = x - y, \qquad \dot{y} = x^2 - 4$

f.p.'s $\rightarrow x^* = (\pm 2, \pm 2)$ (i.e. there are two.)

nullclines : $\begin{cases} \dot{x} = 0 & y = x \\ \dot{y} = 0 & x = \pm 2. \end{cases}$

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[From a nullcline analysis one can predict spiral @ (2,2), saddle @ (−2,−2).]

let $\quad x = x^* + u \quad , \quad y = y^* + v$

$\dot{x} = \dot{u} \quad , \quad \dot{y} = \dot{v}$

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} \partial\dot{x}/\partial x \,|_{x^*,y^*} & \partial\dot{x}/\partial y \,|_{x^*,y^*} \\ \partial\dot{y}/\partial x \,|_{x^*,y^*} & \partial\dot{y}/\partial y \,|_{x^*,y^*} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 2x^* & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

$\underline{\underline{a}}$ $(2,2)$: $\quad J = \begin{pmatrix} 1 & -1 \\ 4 & 0 \end{pmatrix}$

eigenvalues $\rightarrow \quad (1-\lambda)(-\lambda)+4=0 \quad \rightarrow \quad \boxed{\lambda_{\pm} = \frac{1}{2} \pm \frac{i}{2}\sqrt{15}}$

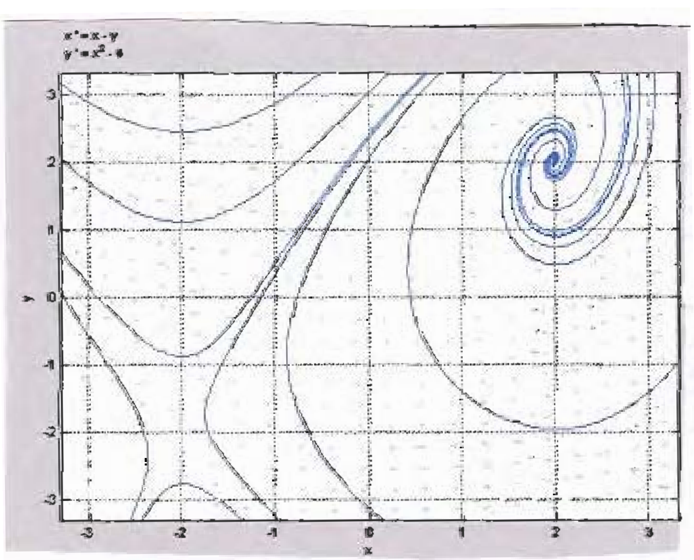
thus $(2,2)$ is an <u>UNSTABLE SPIRAL.</u>

a plot shows the trajectories flow C<u>C</u>W.

$\underline{\underline{b}}$. $(-2,-2)$: $\quad J = \begin{pmatrix} 1 & -1 \\ -4 & 0 \end{pmatrix}$

eigenvalues $\rightarrow \quad (1-\lambda)(-\lambda)-4=0 \quad \Rightarrow \quad \boxed{\lambda_{\pm} = \frac{1}{2} \pm \frac{1}{2}\sqrt{17}}$

thus $(-2,-2)$ is a <u>saddle node</u>



- $\dot{x} = \sin y, \quad \dot{y} = \cos x$

fixed points whenever $\sin y = 0$ and $\cos x = 0$

i.e. @ $(x_m^*, y_n^*) = \left(\left(\tfrac{1}{2} + m \right) \pi, \, n\pi \right)$

$(n, m) \in \mathbb{Z}^2$

linearization gives

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} 0 & \cos y^* \\ -\sin x^* & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & (-1)^n \\ (-1)^{m+1} & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

$\Rightarrow$ eigenvalues: $\lambda_{\pm} = \begin{cases} \pm 1 & n+m \in \text{odds} \\ \pm i & n+m \in \text{evens} \end{cases}$

The former case gives saddle nodes.
The latter case gives centers.

We may also solve for the eigenvectors in the former case:

One gets $\begin{pmatrix} \bar{u} \\ \bar{v} \end{pmatrix} = \begin{pmatrix} 1 \\ \mp 1 \end{pmatrix} \rightarrow n$ odd, m even.

$\begin{pmatrix} \bar{u} \\ \bar{v} \end{pmatrix} = \begin{pmatrix} 1 \\ \pm 1 \end{pmatrix} \rightarrow n$ even, m odd.

Furthermore the system is integrable, incidentally:

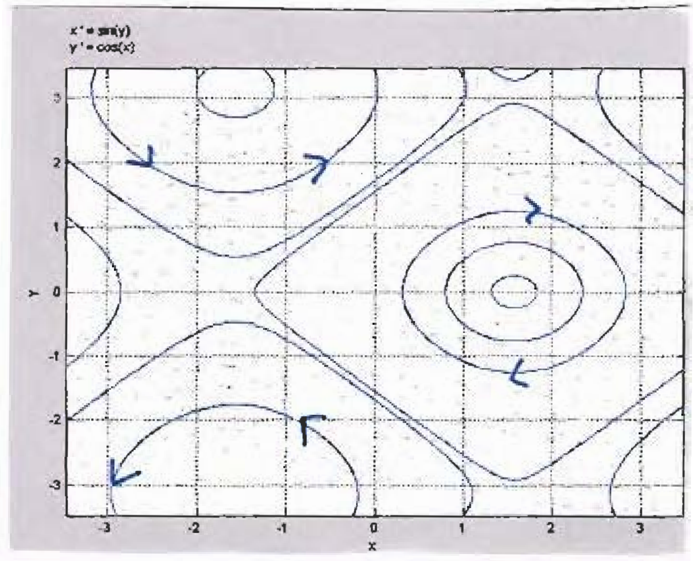
$\psi(x, y) = \sin x + \cos y = C$ = arbitrary constant.

linearizing about (x_m^*, y_n^*) yields

$$C = (-1)^m (x - x_m^*)^2 + (-1)^n (y - y_n^*)^2$$

which prove to be hyperbolas for $n+m \in$ odds
circles $n+m \in$ evens.

This proves the centers are robust, as does reversibility
(see Strogatz p. 164-5 for more info. on the latter)



centers flow CW for n, m even

CCW for n, m odd.

• $x' = xy - 1$; $y' = x - y^3$

fixed points at $(\pm 1, \pm 1)$

null clines → $\begin{cases} \dot{x} = 0 & xy = 1 \\ \dot{y} = 0 & y = x^{1/3} \end{cases}$

from the latter, we can guess $(1,1)$ is a saddle and $(-1,-1)$ is a stable spiral.

Now, $\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} y^* & x^* \\ 1 & -3y^{*2} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$ (after linearization)

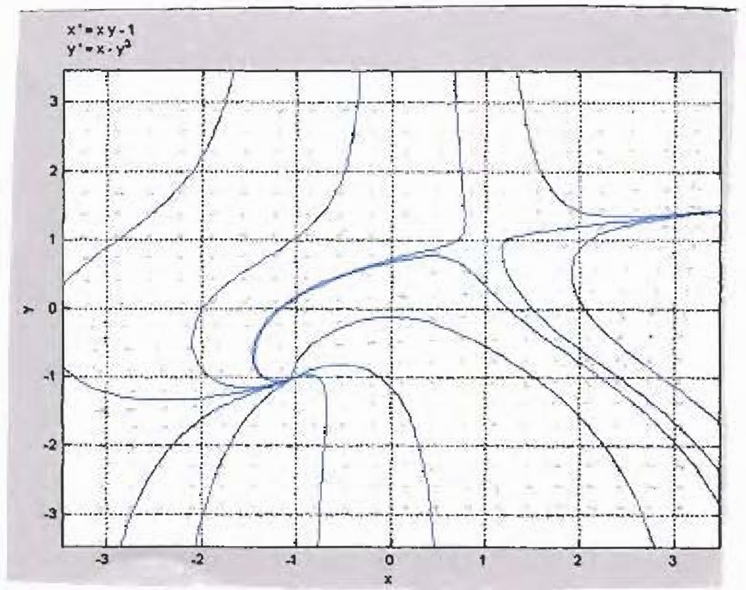
a. $(1,1)$ → $J = \begin{pmatrix} 1 & 1 \\ 1 & -3 \end{pmatrix}$ $\lambda_\pm = -1 \pm \sqrt{5} \gtrless 0$

⇒ SADDLE.

eigenvectors : $\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 2 \pm \sqrt{5} \\ 1 \end{pmatrix}$

b: $(-1,-1)$ → $J = \begin{pmatrix} -1 & -1 \\ 1 & -3 \end{pmatrix}$ $\lambda_\pm = -2$

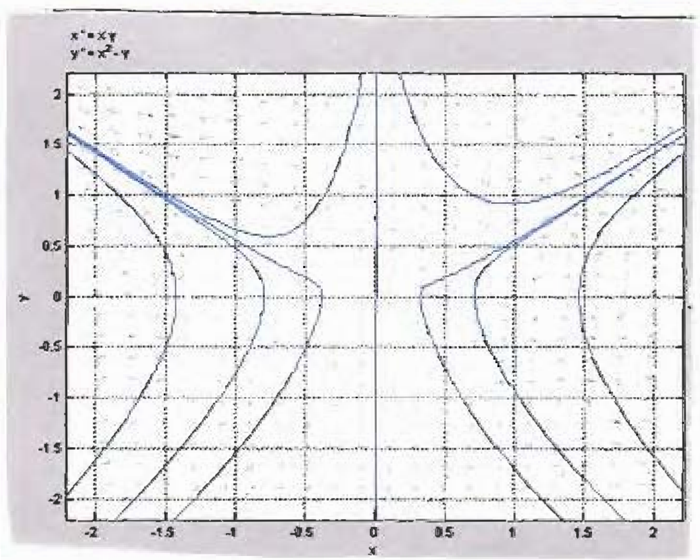
thus linearization predicts a stable star node with one eigendirection $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.



- $\dot{x} = xy$, $\dot{y} = x^2 - y$

There is a single f.p @ $(0,0)$. We will observe that linearization <u>fails</u> because it does not take the unstabilizing effect of the x^2 term into account.

A numerical analysis of the problem shows we have something to the effect of a SADDLE:



One can observe that the stable manifold is $x = 0$ and that far from the origin, the unstable manifold is $y = |x|$. Both are verifiable using the dynamical equations.

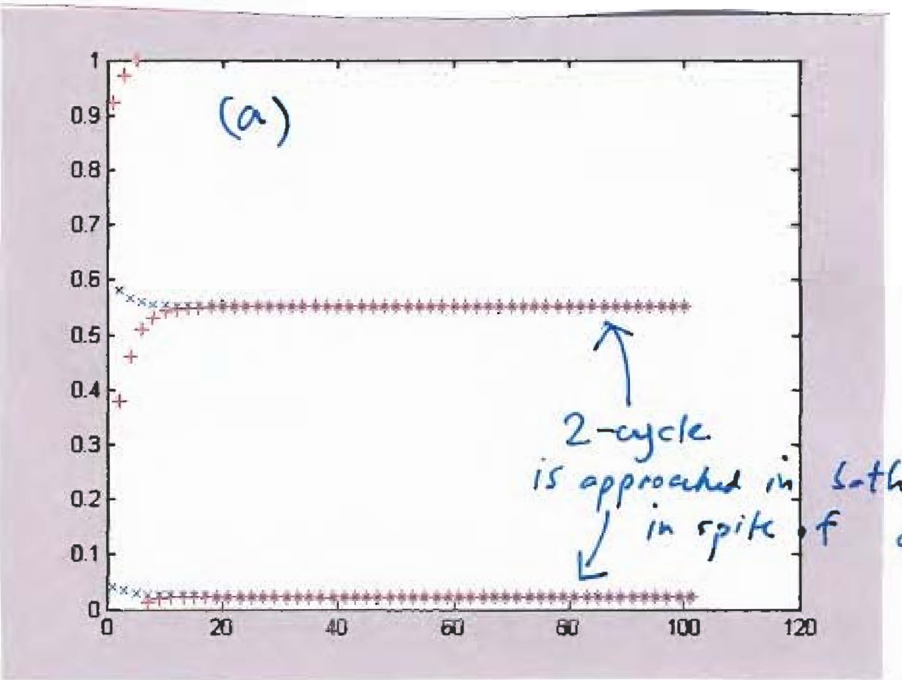
However, linearization gives:

$$\begin{pmatrix}\dot u\\ \dot v\end{pmatrix} = \begin{pmatrix}0 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}u\\ v\end{pmatrix} \Rightarrow \lambda = \{0,-1\}$$

which is the boundary case of a non-isolated
fixed point. The x^2 term however destabilizes the
x-axis and there are therefore not a continuous
line of stable fixed points along x-axis as linear-
ization would suggest.

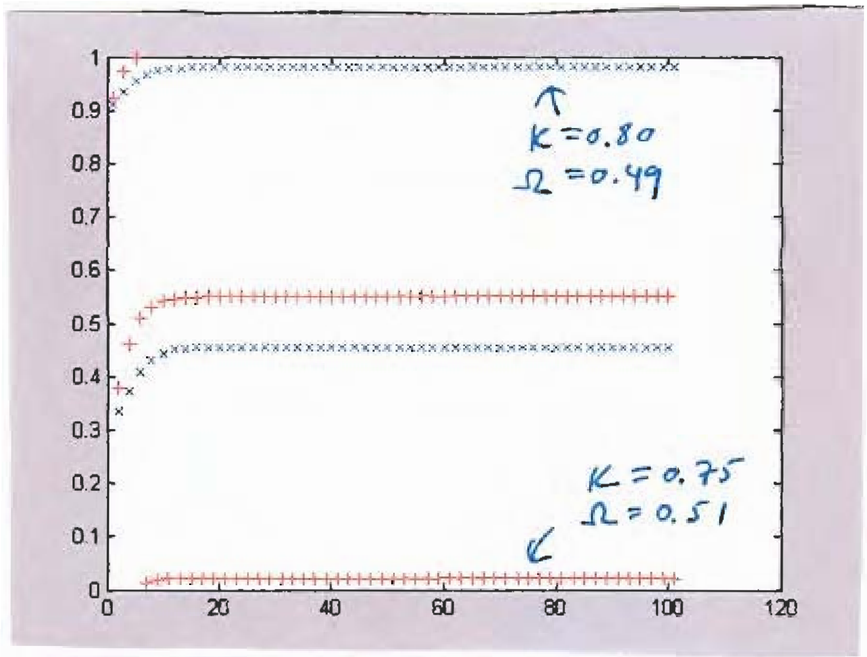
(c) Circle Map

$$\theta_{n+1} = \theta_n + \Omega - \frac{k}{2\pi}\sin(2\pi\theta_n) \mod 1.$$



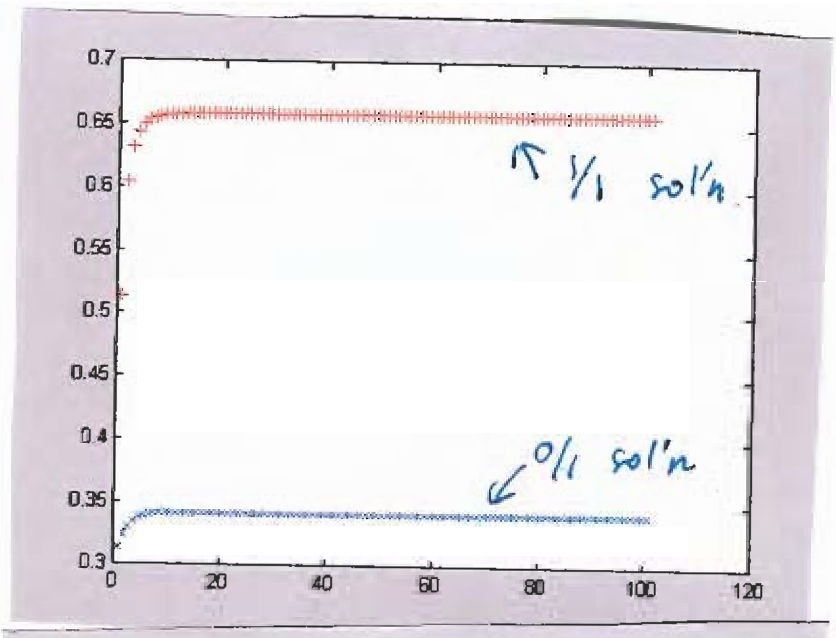
2-cycle is approached in both cases, in spite of different I.C.'s

(b)



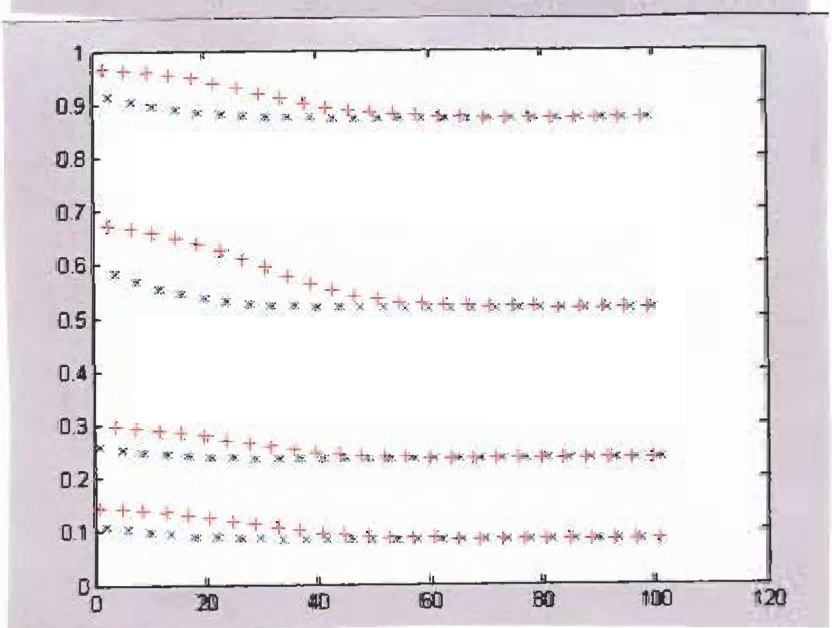
IC = 0.3001 in both trials. Hence the 2-cycle depends on (K, Ω) but not on IC.

(2)



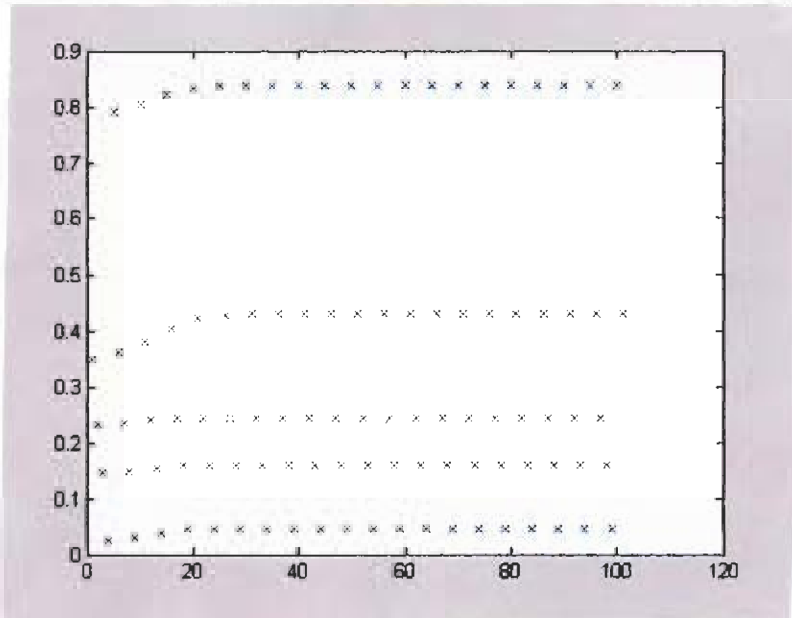
In these trials, the 1/1 solution has $K = 0.75$ and $\Omega = 0.9$. The 0/1 solution has $K = 0.75$ and $\Omega = 0.1$. The latter is stationary on the circle while the former travels once around the circle per iteration.

(3)



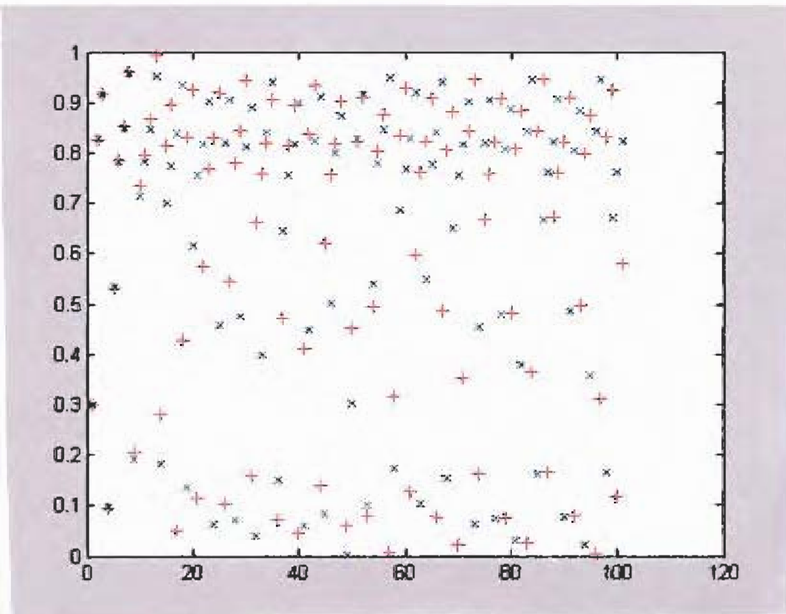
An attracting 3/4-orbit. IC's are 0.3 and 0.6. Note the orbit travels around the circle 3 times in one full period (period = 4 iterations.)

We may observe periodic solutions for $K > 1$:



$K = 1.1$

$\Omega = 0.75$



$K = 1.5$

$\Omega = 0.3$

$IC = 0.300, 0.301$

Notice how quickly these solutions diverge from eachother. This is an example of chaos.