(e) Linearized 2d systems

for any system \( \dot{x} = Ax \), \( \dot{y} = 0 \) is a fixed point. It is the only fixed point if \( A \) is not singular. None of the following \( A \)'s are singular, thus the origin is the unique f.p.: 

\[ \begin{align*}
\dot{x} &= x - y \\
\dot{y} &= x + y
\end{align*} \]

matrix form: 
\[
\begin{pmatrix}
\dot{x} \\
\dot{y}
\end{pmatrix} = 
\begin{pmatrix}
1 & -1 \\
1 & 1
\end{pmatrix}
\begin{pmatrix}
x \\
y
\end{pmatrix}
\]

eigenvalues \( \lambda = \pm i \) 

eigenvectors \( \begin{pmatrix} \pm i \\ \mp i \end{pmatrix} \)

The real part of the eigenvalue complex conjugate pair tells us the origin is UNSTABLE. The imaginary part gives oscillations \( \Rightarrow \) SPIRAL.

A plot of the trajectories shows the spiral is C.C.W.
\[ \dot{x} = 5x + 2y, \quad \dot{y} = -17x - 5y \]

Matrix form: \[
\begin{pmatrix}
\dot{x} \\
\dot{y}
\end{pmatrix} =
\begin{pmatrix}
5 & 2 \\
-17 & -5
\end{pmatrix}
\begin{pmatrix}
x \\
y
\end{pmatrix}
\]

Eigenvalues \( \lambda \) \( \lambda = 8 \pm 3i \)

Thus, the origin is a CENTER. There are no destabilizing nonlinear terms. The trajectories can be shown to be ellipses with semi-major axis tilted from the y-axis at angle \( \theta = 16.84^\circ \) \( = \frac{1}{2} \tan^{-1}(\frac{3}{8}) \).

(See Fig.) Plot shows trajectories flow CW.

Eigenvalues:

\[
\begin{pmatrix}
x' \\
y'
\end{pmatrix} =
\begin{pmatrix}
-5 + 3i \\
17
\end{pmatrix}
\]
\( \dot{x} = 5x + 10y \quad \dot{y} = -x - y \)

Matrix form:
\[
\begin{pmatrix}
\dot{x} \\
\dot{y}
\end{pmatrix}
= 
\begin{pmatrix}
5 & 10 \\
-1 & -1
\end{pmatrix}
\begin{pmatrix}
x \\
y
\end{pmatrix}
\]

Eigenvalues:
\((5 - \lambda)(-1 - \lambda) + 10 = 0 \Rightarrow \lambda = 2 \pm i\)

Real part \(\lambda = 0\) (unstable), imaginary part \(i\) (spiral)

Eigenvectors:
\[
\begin{pmatrix}
x \\
y
\end{pmatrix}
= 
\begin{pmatrix}
-2 + i \\
1
\end{pmatrix}
\]

Plots show trajectories moving CW.

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(5) **Nonlinear 2D Systems**

\( \dot{x} = x - y \quad \dot{y} = x^2 - 4 \)

Fixed points \(\mathbf{x}^* = (\pm 2, \pm 2)\) (i.e., there are two.)

Nullclines:
\[
\begin{cases}
\dot{x} = 0 & \Rightarrow y = 0 \\
\dot{y} = 0 & \Rightarrow x = \pm 2.
\end{cases}
\]

[From a nullcline analysis one can predict spiral @ (2,2), saddle @ (-2, 2)]
\[ x' = -x + u, \quad y' = y + u \]

\[ x = u, \quad y = v \]

\[ (v) = (\frac{\partial f}{\partial x}(x, y), \quad \frac{\partial f}{\partial y}(x, y)) (u) = \begin{pmatrix} 1 & -1 \\ 2x_0 & 0 \end{pmatrix} (u) \]

\[ (2, 2) : \quad J = \begin{pmatrix} 1 & -1 \\ 4 & 0 \end{pmatrix} \]

Eigenvalues: \( (1+\sqrt{15}) \) and \( (1-\sqrt{15}) \)

Thus, \((2, 2)\) is an **unstable spiral**.

A plot shows the trajectories flow CCW.

\[ (-2, -2) : \quad J = \begin{pmatrix} 1 & -1 \\ -4 & 0 \end{pmatrix} \]

Eigenvalues: \( (-1+\sqrt{17}) \) and \( (-1-\sqrt{17}) \)

Thus, \((-2, -2)\) is a **saddle node**.
\[ x = \sin y, \quad y = \cos x \]

Fixed points whenever \( \sin y = 0 \) and \( \cos y = 0 \)
i.e. \[ (x^n, y^n) = ((\frac{1}{2} + m) \pi, n\pi) \]
\((n, m) \in \mathbb{Z}^2\)

Linearization gives
\[ \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & \cos y^n \\ -\sin x^n & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & (-1)^n \\ -(-1)^n & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \]

- \( \lambda_+ = \frac{1 \pm \sqrt{1 + 4(-1)^n}}{2} \) when \( n \) odd.
- \( \lambda_- = \frac{1 \pm \sqrt{1 - 4(-1)^n}}{2} \) when \( n \) even.

The former case gives saddle nodes.
The latter case gives centers.

We may also order the eigenvectors in the former case:
One gets
\[ \begin{pmatrix} 1 \\ (-1)^n \end{pmatrix} \rightarrow \text{ } n \text{ odd, } m \text{ even.} \]
\[ \begin{pmatrix} 1 \\ (-1)^n \end{pmatrix} \rightarrow \text{ } n \text{ even, } m \text{ odd.} \]

Furthermore, the system is integrable, invariants:
\[ y(x, y) = \sin x + \cos y = C = \text{arbitrary constant.} \]

Linearizing about \( (x^n, y^n) \) yields
\[ C = (-1)^n (x-x^n)^2 + (-1)^n (y-y^n)^2 \]

Which proves to be hyperbolas for \( n \) odd, 

circles for \( n \) even.

This proves the centers are robust, as does reversibility

(See Strogatz p. 184-5 for more info on the latter).
\[ x' = y - 1; \quad y' = x - y^3 \]

**fixed points** \( \pm (\pm 1, \pm 1) \)

nullclines \( \implies \) \[ \begin{cases} x = 0 \quad ry = 1 \\
y = 0 \quad y = x^{1/3} \end{cases} \]

from the latter, we can guess \((1,1)\) is a saddle and \((-1,-1)\) is a stable spiral.

\[ (\mathbf{w}) = \begin{pmatrix} y^2 & x^2 \\ 1 & -2y^2 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \]

\( \mathbf{A} : (1,1) \rightarrow \mathbf{J} = \begin{pmatrix} 1 & -1 \\ 1 & -3 \end{pmatrix} \quad \lambda_1 = -1 \pm \sqrt{2} < 0 \quad \text{(SADDLE)} \)

\( \mathbf{v} \quad \text{eigenvectors} \quad \begin{pmatrix} \mathbf{v} \\ \mathbf{v} \end{pmatrix} = \begin{pmatrix} 2 \pm \sqrt{2} \\ 1 \end{pmatrix} \)

\( \mathbf{b} : (-1,-1) \rightarrow \mathbf{J} = \begin{pmatrix} -1 & -1 \\ 1 & -3 \end{pmatrix} \quad \lambda_2 = -2 \)

thus linearisation predicts a stable star node with eigendirection \( \begin{pmatrix} 1 \\ 1 \end{pmatrix} \)
\[
\dot{x} = x y, \quad \dot{y} = x^2 - y.
\]

There is a single fixed point @ (0,0). We will observe that linearization fails because it does not take the destabilizing effect of the \(x^2\) term into account.

A numerical analysis of the problem shows we have something to the effect of a FALLOUT:

One can observe that the stable manifold is \(x = 0\) and that far from the origin, the unstable manifold is \(y = |x|\), both are verifiable using the dynamical equations.
However, linearization gives:

\[
\begin{pmatrix}
  \dot{u} \\
  \dot{v}
\end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}
\begin{pmatrix}
  u \\
  v
\end{pmatrix} = \lambda \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}
\]

which is the boundary case of a non-isolated fixed point. The $y^2$ term however destabilizes the $x$-axis and there are therefore not a continuous line of stable fixed points along $x$-axis as linearization would suggest.

\[(C)\]

Circle Map

\[\theta_{n+1} = \theta_n + \Delta - \frac{k}{2\pi} \sin(2\pi \theta) \mod 1.\]
We may observe periodic solutions for \( k > 1 \):

\[
\begin{align*}

&k = 1.1 \\
&\ell = 0.2 \\
&\text{or } k = 0.3, 0.31
\end{align*}
\]

Noting how quickly the solutions diverge from each other. This is an example of chaos.