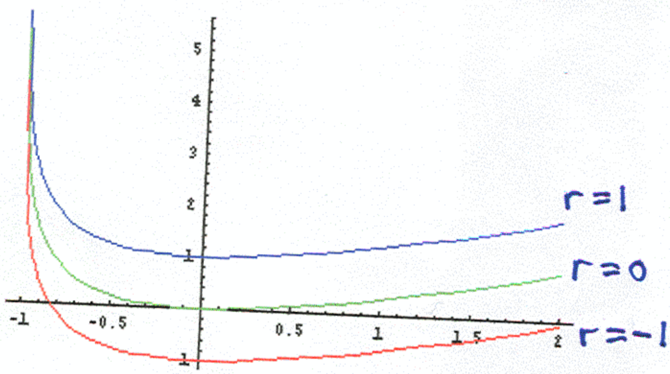
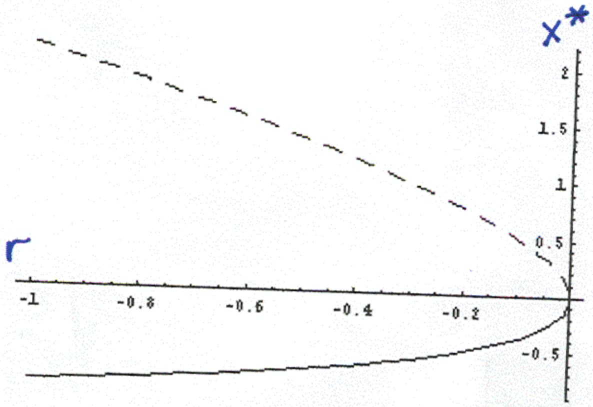
APM 203 Homework Assignment #2

Solutions

1. (i) $\dot{x} = r + x - \ln(1+x)$

```
Plot[{f[-1, x], f[0, x], f[1, x]}, {x, -1, 2}, PlotStyle ->
  {RGBColor[1, 0, 0], RGBColor[0, 1, 0], RGBColor[0, 0, 1]}]
```





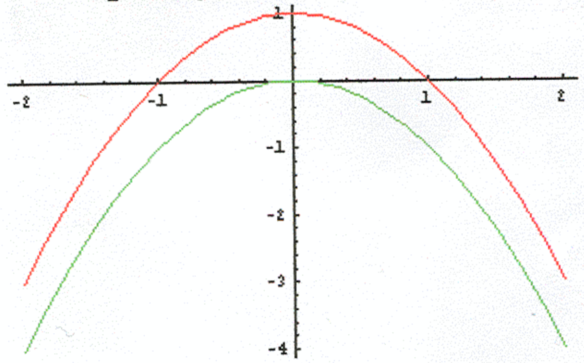
$$f(x) = r + x - \ln(1+x) \approx r + x - \left(x - \frac{x^2}{2}\right)$$

$$= r + \frac{x^2}{2} \quad \Rightarrow \quad \text{saddle-node bif.}$$

(ii) $\dot{x} = r^2 - x^2$

$$f(x) = r^2 - x^2 = 0 \Rightarrow x^* = \pm r$$

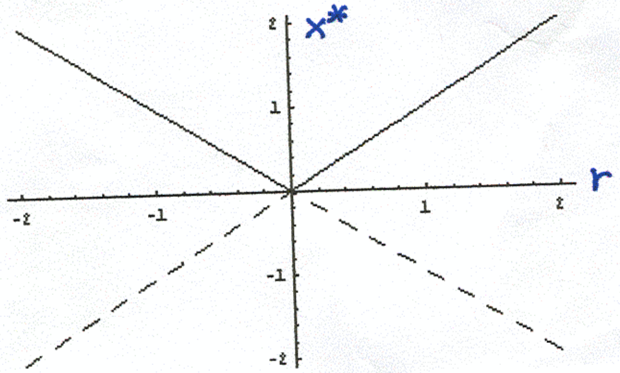
```
Plot[{1-x^2, -x^2}, {x, -2, 2},
 PlotStyle -> {RGBColor[1, 0, 0], RGBColor[0, 1, 0]}]
```



Parabolas dip to the x-axis without crossing (thus, not in fact creating a saddle-node bifurcation)

The (x^*, r)-bifurcation plot looks thus:

```
Plot[{Abs[r], -Abs[r]}, {r, -2, 2},
 PlotStyle -> {GrayLevel[0], Dashing[{.03}]}]
```



Doesn't look familiar? Try transformation (linear)

$y = x + r \Rightarrow \dot{x} = r^2 - x^2 = (r - x)(r + x) = y(2r - y)$.

Aha! This is in fact a transcritical bifurcation

(Not two saddle-nodes in one!)

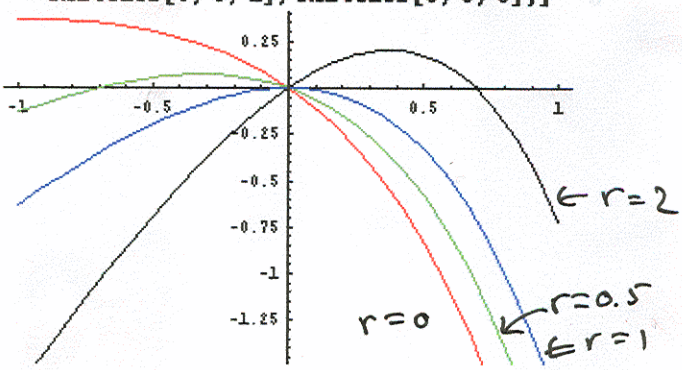
(iii) $\dot{x} = x(r - e^{x})$ $\qquad (x^*, r^*) = (0, 1)$ – bifurcation point.

$$\cong x\left(r - 1 - x - \frac{x^2}{2}\right) = x(r-1) - x^2 + O(x^3)$$

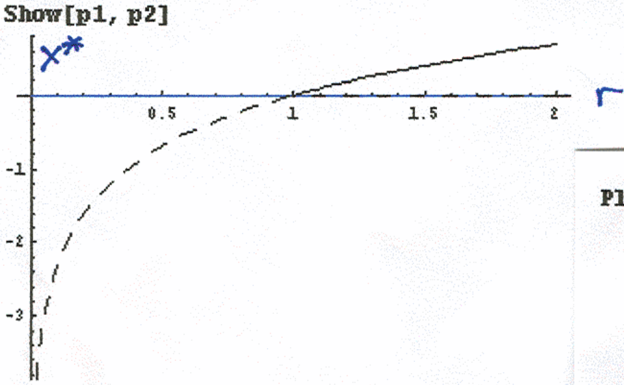
recasting $r = r' + 1 \Rightarrow \dot{x} = r'x - x^2 = x(r' - x)$

a transcritical system!

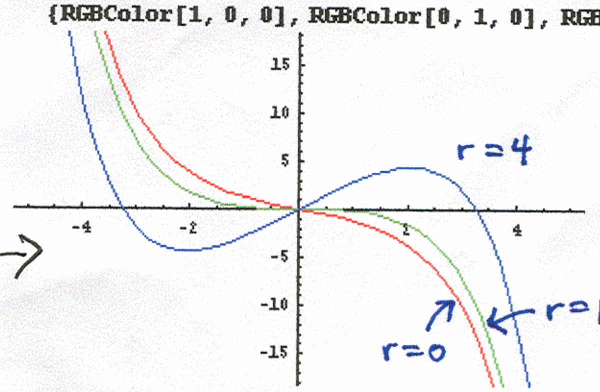
```
Plot[{f[0, x], f[0.5, x], f[1, x], f[2, x]}, {x, -1, 1},
 PlotStyle -> {RGBColor[1, 0, 0], RGBColor[0, 1, 0],
   RGBColor[0, 0, 1], RGBColor[0, 0, 0]}]
```




```
p1 = Plot[{0, Log[r]}, {r, 0, 1}, PlotStyle ->
   {RGBColor[0, 0, 1], {Dashing[{.03}], RGBColor[0, 0, 0]}}]
p2 = Plot[{0, Log[r]}, {r, 1, 2}, PlotStyle ->
   {{Dashing[{.03}], RGBColor[0, 0, 1]}, RGBColor[0, 0, 0]}]
Show[p1, p2]
```




```
Plot[{f[0, x], f[1, x], f[4, x]}, {x, -5, 5}, PlotStyle ->
 {RGBColor[1, 0, 0], RGBColor[0, 1, 0], RGBColor[0, 0, 1]}]
```

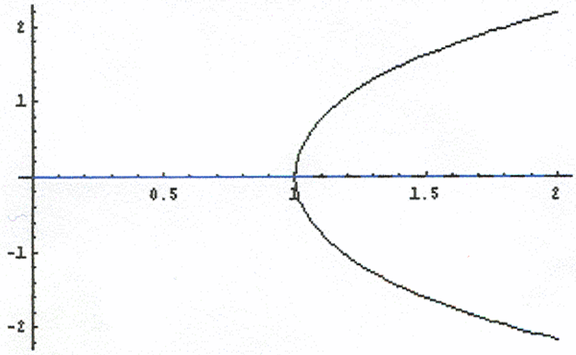


(iv) $\dot{x} = rx - \sinh x$

$$\cong (1 + r')x - \left(x + \frac{x^3}{3!}\right)$$

$$= r'x - \frac{x^3}{6} \Rightarrow \text{pitchfork (supercritical)}$$

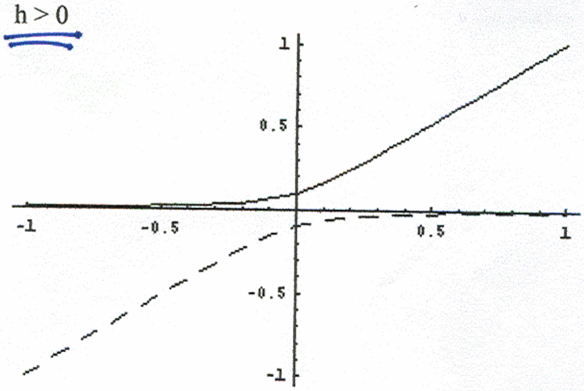
```
p1 = Plot[{x /. FindRoot[f[r, x], {x, 2}],
    0, x /. FindRoot[f[r, x], {x, -2}]},
  {r, 1, 2}, PlotStyle -> {RGBColor[0, 0, 0],
    {Dashing[{.03}], RGBColor[0, 0, 1]}, RGBColor[0, 0, 0]}]
p2 = Plot[0, {r, 0, 1}, PlotStyle -> RGBColor[0, 0, 1]]
Show[p1, p2]
```



2.

(i)

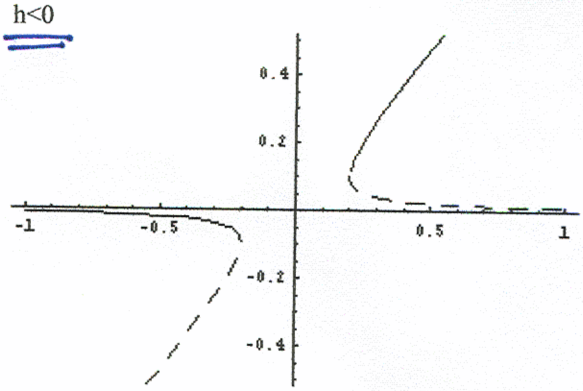
h > 0



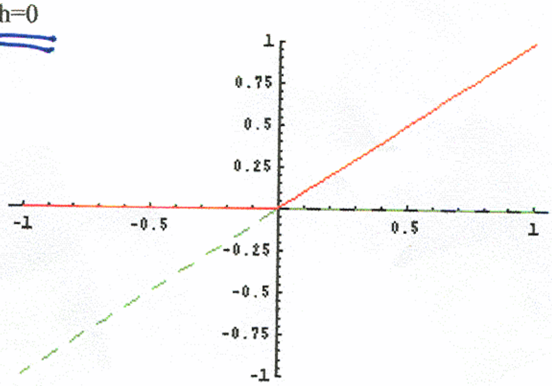
$$\dot{x} = h + rx - x^2$$

$$x^*(r) = \frac{1}{2}\left(r \pm \sqrt{r^2 + 4h}\right)$$

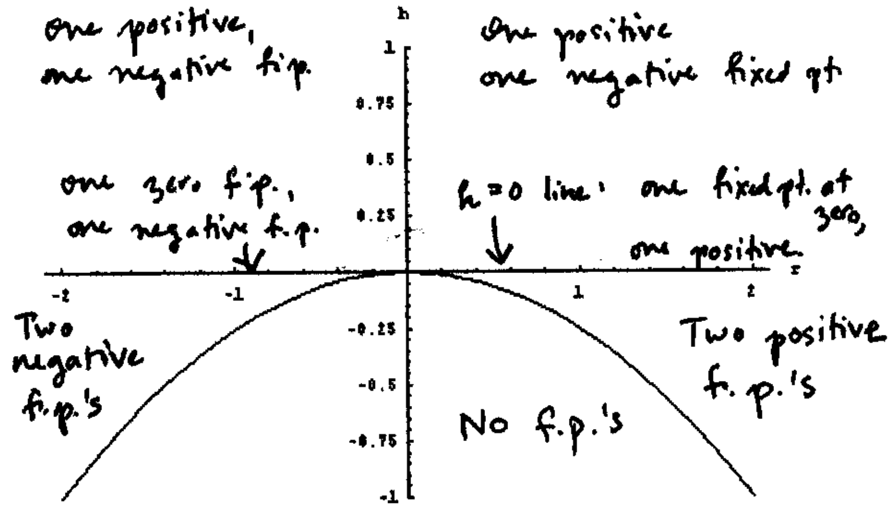
h<0



h=0



(ii)

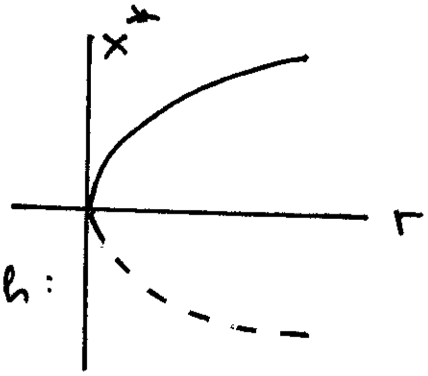


on the $h=0$ line, a transcritical bifurcation occurs (at $r=0$)

and on the $h=-\frac{1}{4}r^2$ parabola, saddle-node bifurcations on either side.

(iii) $\dot{x} = r - x^2$

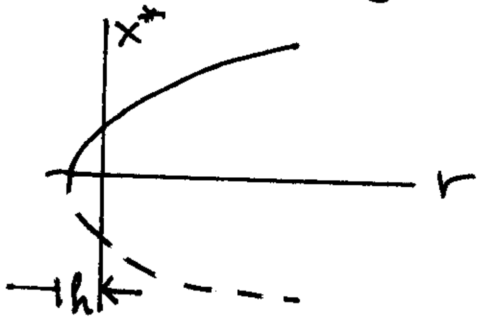
bifurcation diagram without perturbation, h:



with perturbation the bifurcation diagram looks identical,

$\dot{x} = h + r - x^2$

(Indeed, it looks identical after perturbation hx, as well.)



simply shifted by h. This is because we're essentially resetting r to some value $r + \Delta r$. Behavior should not change.

5

Numerics : bonus question

3. Advection Eqn.

$$\frac{\partial F}{\partial t} + c\frac{\partial F}{\partial x} = 0 \quad ; \qquad c > 0.$$

Discretize :

$$\frac{F_{m,n+1} - F_{m,n-1}}{2\Delta t} = -c\,\frac{F_{m+1,n} - F_{m-1,n}}{2\Delta x}$$

Making the ansatz $F_{m,n} = B^{n\Delta t} e^{i\mu m\Delta x}$, we get :

$$i \cdot \frac{B^{\Delta t} - B^{-\Delta t}}{2} = \sigma \quad . \quad \text{Let} \quad x \equiv B^{\Delta t},$$

$$x^2 + 2i\sigma x - 1 = 0 \quad \Rightarrow \quad x_\pm = B^{\Delta t} = -i\sigma \pm \sqrt{1-\sigma^2}$$

if $|x| > 1$, the solution grows exponentially, now

$$|x_+| = \left\{(\mathrm{Re}\, x_+)^2 + (\mathrm{Im}\, x_+)^2\right\}^{1/2} = \begin{cases} 1 & |\sigma| < 1 \\ -\sqrt{\sigma^2-1} + \sigma & |\sigma| > 1 \end{cases}$$

The other solution, x_- dies off (transient). Thus, for $|\sigma| > 1$, the numerical solution diverges. Therefore for later times, the amplitude will be artifically large. This is not true behavior (for real c.)

$\boxed{6}$