1. \( x' = 1 - e^{-x^2} \equiv f(x) \)

**fixed points:** \( f(x^*) = 0 \Rightarrow 1 - e^{-x^*^2} = 0 \)

or \( x^*^2 = 0 \), i.e., \( x^* = 0 \).

**stability:** \( x = x^* + \delta x = \delta x \)

\[ f(x) = 1 - e^{-\delta x^2} \approx 1 - (1 - \delta x^2) = \delta x^2 \]

= \( o(\delta x^2) \)

\[ \Rightarrow \text{linearization fails.} \]

**Graphical approach:**

\( x^* = 0 \) is semi-stable (stable on the left but unstable on the right.)
2. $\dot{x} = ax - x^3$

**fixed points:**
- $a < 0 \Rightarrow x^* = 0$
- $a = 0 \Rightarrow x^* = 0$
- $a > 0 \Rightarrow x^* = 0, x^* = \pm \sqrt{a}$

we may recognize this system as having a

- pitch fork bifurcation

at $a = 0$:

**stability:**
- $a < 0$, $x = x^* + \delta x = \delta x$
  \[ \dot{x} = a \delta x \Rightarrow \text{stable (a < 0)} \]

next, $a = 0$, $x = x^2 + \delta x = \delta x$

\[ \dot{x} = -\delta x^3 = O(\delta x)^3 \Rightarrow \text{linearization fails.} \]

Graph:

next, $a > 0$. Now we have 3 h.p.'s

- $x^* = 0$: $\dot{x} = x(a - x^2) \equiv a \delta x \Rightarrow \text{unstable}$
- $x^* = \pm \sqrt{a}$: $\dot{x} \equiv (\pm \sqrt{a} + \delta x)(\mp 2\sqrt{a} \delta x) \equiv -2a \delta x \Rightarrow \text{stable}$
3. \( \dot{x} = x(1-x)(2-x) = f(x) \)

\[
\text{f.p.'s : } \quad f(x^*) = 0 \implies x^* = 0, 1 \text{ or } 2.
\]

\underline{Stability:}

(i) \( x^* = 0 \) : \( x = x^* + \delta x = \delta x \)

\[
f(x^* + \delta x) = \delta x (1 - \delta x)(2 - \delta x) \approx 2 \delta x \implies \text{unstable.}
\]

(ii) \( x^* = 1 \) : \( x = x^* + \delta x = 1 + \delta x \)

\[
f(x^* + \delta x) = (1 + \delta x)(-\delta x)(2 - \delta x - 1) \approx -\delta x \implies \text{stable.}
\]

(iii) \( x^* = 2 \) : \( x = x^* + \delta x = 2 + \delta x \)

\[
f(x^* + \delta x) = (2 + \delta x)(1 - \delta x)(-\delta x) = -2 \delta x \implies \text{unstable.}
\]

4. \( \dot{x} = x^2 (6-x) \) \( \text{f.p.'s : } \quad x^* = 0 \text{ & } x^* = 6. \)

Linearization fails for \( x^* = 0. \) Graph:

\[
\Rightarrow \text{uni-stable.}
\]

Next, \( x^* = 6 \) : \( x = 6 + \delta x \implies \dot{x} = (36 + 12 \delta x)(-\delta x) \leq -36 \delta x \\
\text{i.e. stable.} \]
5. \[ \dot{x} = \ln x \quad \Rightarrow \quad f(x^*) = \ln x^* = 0 \quad \Rightarrow \quad x^* = 1 \]

Linearization: \[ x = 1 + \delta x \quad \Rightarrow \quad \dot{x} = \ln (1 + \delta x) \approx \delta x \]

i.e. unstable

(B) \[ x_{n+1} = f(x_n) \quad f(x) = rx(1-x) \]

\[ \chi \in [0,1] \quad \forall n \Rightarrow r > 0, \text{ otherwise} \]

\[ [0,1] \not\Rightarrow \text{negative } \chi. \]

Furthermore, \[ f(\frac{1}{2}) = \frac{r}{4} = \max \{ f([0,1]) \} \]

\[ \Rightarrow r < 4. \quad \text{Thus } 0 \leq r \leq 4. \]

Now, we find dissipative value of r. The latter requires \[ |f'(x)| < 1 \quad \text{everywhere on the interval.} \]

Thus \[ |r(1-2x)| < 1. \quad \text{L.H.S. maximizes at } x = 0 \& 1. \]

so we need \[ |r(1-2 \cdot 0)| = |r(1-2 \cdot 1)| < 1 \quad \text{i.e. } r < 1 \]

using \( r > 0 \).

Finally, \[ 0 < r < 1. \]
2. What is the asymptotic behavior for large $n$? Do this analytically first:

$$x_{n+1} = f(x_n) = rx_n(1-x_n) \quad 0 \leq r < 1.$$ 

since $0 \leq x_n \leq 1$ we therefore have $0 \leq x_{n+1} < x_n$

stronger inequality: $0 \leq x_{n+1} < rx_n$

$$\Rightarrow 0 \leq x_{n+2} < rx_{n+1} < r^2x_n \ldots \text{ad infinitum}$$

i.e. $0 \leq x_n < r^n x_0$ but $0 \leq r < 1$

thus R.H.S. $\rightarrow 0$ and $x_n$ is sent to $0$.

Graphically (we cobwebs!)

![Graph showing cobwebbing method to demonstrate the convergence to zero](image_url)
3. Fixed points and their stability:

Investigate ranges of \( r \) within \( 0 \leq r \leq 4 \).

(i) First, let \( 0 \leq r < 1 \). From part 2,

we expect 1 stable eq. \((x = 0)\).

Verify this again: \( x^n = f(x^n) = r x^n (1-x^n) \)

\( \Rightarrow x^* = 0 \) (alternate eq. \( 1 - \frac{1}{r} \leq x, x \leq \frac{1}{r} \)).

For stability, linearize:

\[ \delta x_{n+1} = f'(x^*) \delta x_n = r \delta x_n \]

\( \Rightarrow \) **STABLE**

(ii) \( r = 1 \). Stability analysis fails. Now, \( x_{n+1} = x_n (1-x_n) \)

if \( x_0 = 0 \) \( \Rightarrow x_1 = 0 \) ... etc.

just away from \( x_0 = 0 \) we have

\[ x_1 = x_0 (1-x_0) \]

\[ x_2 = x_0 (1-x_0) \left[ 1 - x_0 (1-x_0) \right] = x_0 (1-x_0) (1-x_1) \]

\[ \vdots \]

\[ x_n = x_0 (1-x_0) (1-x_1) \cdots (1-x_{n-1}) \Rightarrow 0 \]

\( \Rightarrow \) **stable**

also, using cobweb:
(iii) Let \( 1 < r \leq 4 \), fixed points \( x^* = 0, 1 - \frac{1}{r} \)

stability of \( x^* = 0 \):
\[
\delta x_{n+1} = r \delta x_n = \text{unstable}
\]

of \( x^* = 1 - \frac{1}{r} \):
\[
\delta x_{n+1} = f' \left( 1 - \frac{1}{r} \right) \delta x_n = r \left( 1 - 2 + \frac{2}{r} \right) \delta x_n = (2 - r) \delta x_n
\]

Thus, stable for \( 1 < r < 3 \)
unstable for \( 3 < r \leq 4 \)

and at \( r = 3 \): Investigate behavior of \( f'(f(x)) \).

Thus \( f \left( f \left( \frac{2}{3} + \delta x_n \right) \right) = \frac{2}{3} + \delta x_{n+2} = \frac{2}{3} + \delta x_n - 16\delta x_n^3 - 27\delta x_n^4 \)

\[
\Rightarrow \delta x_{n+2} = \delta x_n \left( 1 - (16\delta x_n^2 - 27\delta x_n^4) \right) < 1
\]

thus \( \frac{2}{3} \) is (marginally) stable for \( r = 3 \).

---

### SUMMARY:

<table>
<thead>
<tr>
<th>( r )</th>
<th>Fixed pts.</th>
<th>Stability</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 0 \leq r &lt; 1 )</td>
<td>0</td>
<td>stable</td>
</tr>
<tr>
<td>( r = 1 )</td>
<td>0</td>
<td>stable (marginally)</td>
</tr>
<tr>
<td>( 1 &lt; r &lt; 3 )</td>
<td>0</td>
<td>1 - ( \frac{1}{r} ) unstable stable</td>
</tr>
<tr>
<td>( r = 3 )</td>
<td>0</td>
<td>2/3 unstable stable (marginally)</td>
</tr>
<tr>
<td>( 3 &lt; r \leq 4 )</td>
<td>0</td>
<td>1 - ( \frac{1}{r} ) unstable unstable</td>
</tr>
</tbody>
</table>
$r = 0.4$

All initial values are mapped to zero as illustrated here (dissipative region?).

Limiting value

Limiting value
Here, we are in the stable region for $r = \frac{3}{2}$. Limiting value

Logistic map

$x(n)$ vs. $x(n+1)$

Logistic map

$x(n)$ vs. $n$

Asymptotic limit
Here, we have oscillating motion to the stable fixed point \( r = 2.9 \).

\[ 1 - \frac{1}{2.9} \approx 0.65 \]
The system enters into a 2-cycle. F.P. $1 - \frac{1}{3.2} \approx 0.69$ is no longer stable.
$r = 3.8$, $x_n$ is filling out a continuous range of values in $[0,1]$. wild, but bounded motion. No periodicity, no asymptotic limit.
Now all values of $x$ in $[0,1]$ are visited, the curve eventually becomes continuous. No periodicity, no limiting value.

Very erratic and ranges from 0 to 1.
Special cases:

$r = 1$, slow damping to zero.

(How we plot every other iteration)

$r = 3$, even slower damping.