NUMERICAL PREDICTION AND DYNAMIC METEOROLOGY

SECOND EDITION

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CHAPTER 5

Numerical Methods

5-1 INTRODUCTION

The previous chapters have presented the fundamental hydrodynamical equations that govern atmospheric motions and also described various types of disturbances, or waves, that characterize these motions—their velocity, pressure and temperature distributions, their growth and decay, and their sources of energy. V. Bjerknes (1904) is recognized as the first to suggest that, given observed initial fields of mass and velocity, it would be possible in principle to determine the mass and velocity distribution at any future time by solving the hydrodynamical equations as an initial value problem. Although these nonlinear partial differential equations do not have, in general, analytic or closed solutions, they can be integrated by numerical methods to yield a forecast of the meteorological variables for a future time. Present numerical weather prediction models, as they are called, show skill over climatological or persistence forecasts for about one week and even longer in some respects.

A pioneering attempt to predict the weather by numerical integration was made by an Englishman, L. F. Richardson (1922), during World War I. Although his procedure was basically sound, there were some flaws that resulted in large errors with respect to the observed fields. Moreover, the enormous computing time required to solve the equations in the general form he used discouraged any immediate further attempts at numerical weather prediction, even to find out why Richardson's forecast went awry. Just after World War II the electronic computer (ENIAC) was installed at Princeton University and Charney, Fjørtoft, and von Neumann (1950) made the first successful numerical forecast at 500 mb with a simple barotropic vorticity model, which was a development by C. G. Rossby in the late 1930s and early 1940s. In fact, there had been considerable progress in both numerical methods and dynamic meteorology between the two wars that made the Princeton numerical forecast more timely than Richardson's effort. This chapter and the next will deal with some numerical methods currently applied to weather prediction.

5-2 FINITE DIFFERENCE METHODS

The most common numerical integration procedure for weather prediction has been the finite-difference method in which the derivatives in the differential

equations of motion are replaced by finite difference approximations at a discrete set of points in space and time. The resulting set of equations, with appropriate restrictions, can then be solved by algebraic methods.

Taylor series may be used to establish appropriate finite difference approximations to derivatives as follows:

$$f(x \pm \Delta x) = f(x) \pm f'(x)\Delta x + f''(x)\frac{\Delta x^2}{2!} \pm f'''(x)\frac{\Delta x^3}{3!} \pm \ldots \quad (5\text{-}1)$$

where for later convenience, $\Delta x > 0$. Using only the $+$ series and solving for $f'(x)$ gives

$$f'(x) = \frac{f(x + \Delta x) - f(x)}{\Delta x} + R \quad (5\text{-}2)$$

where the term of highest order in the remainder R is $-f''(x)\Delta x/2$. When R is dropped in (5-2) the remaining approximation for the derivative $f'(x)$, referred to as a *forward difference* for $\Delta x > 0$, is said to be of *order* Δx, denoted by the symbol $0\ (\Delta x)$. The latter represents the truncation error of the finite difference approximation.

If the series with the negative sign in (5-2) is subtracted from the positive series, the following *centered* difference approximation for $f'(x)$ results:

$$f'(x) = \frac{f(x + \Delta x) - f(x - \Delta x)}{2\Delta x} + O(\Delta x^2) \quad (5\text{-}3)$$

On the other hand, the addition of the two series in (5-1) leads to

$$f''(x) = \frac{f(x + \Delta x) - 2f(x) + f(x - \Delta x)}{\Delta x^2} + O(\Delta x^2) \quad (5\text{-}4)$$

In both cases the truncation error is of order Δx^2 giving more accurate approximations than from (5-2), which might have been expected from geometric considerations.

Since wavelike motions are characteristic of the atmosphere it is of interest to apply the centered difference approximations for $f'(x)$ to a simple harmonic function, $f(x) = A \sin(2\pi x/L)$. The ratio of the finite difference approximation, say $f'_D(x)$ to the true value of $f'(x)$ is readily found to be

$$\frac{f'_D(x)}{f'(x)} = \frac{\sin(2\pi\Delta x/L)}{2\pi\Delta x/L} \quad (5\text{-}5)$$

Since the ratio $(\sin\alpha)/\alpha$ approaches 1 as α approaches 0, it is evident that the finite difference approximation approaches the true value of $f'(x)$ as $\Delta x/L$ approaches zero. Thus the truncation error will be small when Δx is small compared to L. On the other hand, the error can be extremely bad for relatively small L, for example, for $L = 2\Delta x, f'_D(x) = 0$ for all x, regardless of the true value of $f'(x)$.

Figure 5-5 shows an x,t plot of discrete gridpoints. The triangle of large dots represents those (m,n) gridpoints on which the solution for the point (5,3) depends or the *domain of dependence,* or *influence,* when the centered finite difference or *leapfrog* scheme is used on the advection equation. The equation for the left side of the triangle is: $t - (\Delta t/\Delta x)\, x =$ constant. Now suppose that the characteristic through the point (5,3) is the line labeled

$$x - ct = x_0 \qquad \text{(characteristic)}$$

Since it was shown earlier that the solution of the differential equation propagates along characteristics, the true solution at the point (5,3) must be $F(x_0)$. However, in the example illustrated here, the point x_0 lies outside the domain of dependence of the point (5,3); hence the solution obtained from the difference equation cannot give a correct value to the solution of the differential equation. Note that in this case the slope of the triangle side exceeds that of the characteristic, that is,

$$\frac{\Delta t}{\Delta x} > \frac{1}{c} \qquad \text{or} \qquad \frac{c\Delta t}{\Delta x} > 1$$

which violates the computational stability criterion (5-32); so a satisfactory solution to the difference equation cannot be expected.

Another interesting feature of Figure 5-5 is that the lattice of points in the domain of dependence on which the solution to the difference equation for a given point depends consists only of alternate points, except for $t = 0$. This peculiarity can lead to the separation of the solutions at adjacent points, primarily due to the computational mode that "flip-flops" at every time step.

5-4 SOME BASIC CONCEPTS

Having studied the numerical solution to a simple, but important, linear partial differential equation, it is now desirable to discuss some basic, general concepts related to the solution of partial differential equations by numerical methods.

First, if a *difference equation* is substituted for a *differential equation,* intuitively it would seem necessary that if a desirable result is to be obtained, the former should approach the latter as the finite difference increments are reduced to zero.

Definition A difference equation is said to be *consistent* or *compatible* when it approaches the corresponding differential equation as the finite difference increments, Δt, Δx, etc. approach zero.

Since the difference equation is only an approximation to the differential equation, the accuracy of the former may be measured by taking the difference between the two, which is called the *truncation error Tr,*

$$Tr = \text{difference eq} - \text{differential eq} \qquad (5\text{-}33)$$

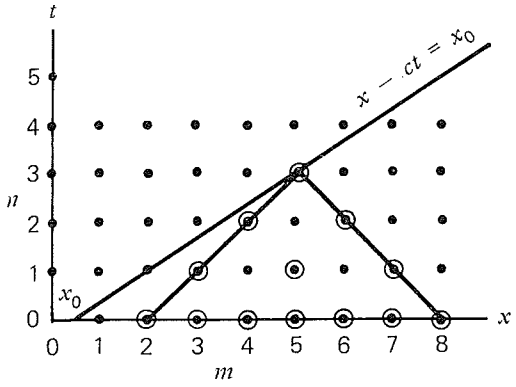

Figure 5.5 Illustrates the domain of dependence of the finite difference solution at $n = 3$ and $m = 5$.

As an example, consider the advection equation previously examined for which

$$Tr = \frac{F_{m,n+1} - F_{m,n-1}}{2\Delta t} + c\frac{F_{m+1,n} - F_{m-1,n}}{2\Delta x} - \left(\frac{\partial F}{\partial t} + c\frac{\partial F}{\partial x}\right)_{m,n} \tag{5-34}$$

To interpret the significance of truncation error, it is necessary to expand the first two terms in Taylor series.

$$F_{m,n\pm1} = F_{m,n} \pm \frac{\partial F}{\partial t}\frac{\Delta t}{1!} + \frac{\partial^2 F}{\partial t^2}\frac{\Delta t^2}{2!} \pm \frac{\partial^3 F}{\partial t^3}\frac{\Delta t^3}{3!} \pm \cdots$$

Similar series can be written for $F_{m\pm1,n}$. When these series are substituted into (5-34), the result is

$$Tr = \frac{\partial^3 F}{\partial t^3}\frac{\Delta t^2}{6} + c\frac{\partial^3 F}{\partial x^3}\frac{\Delta x^2}{6} + \text{higher powers of } \Delta t \text{ and } \Delta x \tag{5-35}$$

or sometimes simply as, $Tr = O(\Delta t^2) + O(\Delta x^2)$. Nevertheless, the size of the coefficients is also important.

It is evident that the difference equation approximation in (5-34) is *consistent* with the differential equation, that is,

$$Tr \to O \text{ as } \Delta t,\ \Delta x \to 0 \tag{5-36}$$

It is quite clear from the previous example that consistency alone is not sufficient to guarantee that the solution of the difference equation will be a good approximation to the solution of the differential equation, which is, after all, the real goal. Again, from intuitive considerations, it would be reasonable to expect that an acceptable solution to the difference equation over some specified interval of space and time would approach the solution of the differential equation as the finite difference increments Δt, Δx, etc., approach zero.

Definition A finite-difference solution is said to be *convergent* if, for a fixed time interval, $T = n\Delta t$, it *approaches the solution of the differential equation* as the increments Δt, Δx, etc. approach zero.

Definition If a *difference scheme* gives a *convergent solution for any initial conditions,* the *scheme is* also said to be *convergent.*

The next definition concerns the stability of a difference system and may be expressed in several ways. In simple terms, any numerical scheme that allows a growth of error that eventually "swamps" the true solution is unstable. The preceding statement is rather vague, however, and a more specific mathematical definition is desirable. Several usable definitions follow.

Definition I According to Richtmyer and Morton (1967) the difference scheme is *stable* if its solutions remain uniformly bounded functions of the initial state for all sufficiently small Δt (that is, $0 < \Delta t \leq \tau$, and $n\,\Delta t = T$).

Definition II When the corresponding differential solution is bounded, a finite difference scheme is *unstable* if, for a fixed spatial grid and homogeneous boundary conditions, there exist initial disturbances for which the finite-difference solution becomes unbounded as n goes to infinity. In general this is a more stringent requirement than Definition I.

Definition III A set of difference equations is *stable* if the cumulative effect of all round-off errors remains negligible as n increases.

The question of convergence is difficult to investigate because it involves derivatives of the true solution for which the upper and lower bounds are not known. On the other hand, stability is generally not as difficult to ascertain; and fortunately, there is an important theorem by Lax that relates consistency and stability to convergence.

Lax Equivalence Theorem (Richtmyer and Morton, 1967)
Given a properly posed, linear initial-value problem and a finite difference approximation that satisfies the *consistency* condition, *stability* (according to Definition I) is the *necessary* and *sufficient* condition for *convergence.*

In the example of the leapfrog scheme applied to the advection equation discussed earlier, the difference equation is consistent and was shown to be stable for $c\Delta t/\Delta x \leq 1$. It was further shown to converge for a particular initial state. On the other hand, when $c\Delta t/\Delta x > 1$, convergence is not possible for all initial states (e.g., all wavelengths).

5-5 STABILITY ANALYSIS

There are several methods of stability analysis. Obviously, when an analytical solution of the difference equation can be found, such as in the application of the leapfrog scheme to the advection equation, the solution can be examined directly. However in many cases other methods are required.

5-5-1 The Matrix Method

In the matrix method the solution of the difference system at every gridpoint at the time $(n + 1)\Delta t$ is assumed to be expressible in terms of the values at time $n\Delta t$ as follows:

$$\mathbf{U}_{n+1} = A\,\mathbf{U}_n \tag{5-37}$$

where $\mathbf{U}$ is a vector representing all of the dependent variables (perhaps augmented) at all of the gridpoints and A is a matrix representing the difference system.

As an example, consider the following parabolic differential and corresponding difference equations

$$\frac{\partial F}{\partial t} = K\frac{\partial^2 F}{\partial x^2} \tag{5-38}$$

and

$$\frac{F_{m,n+1} - F_{m,n}}{\Delta t} = K\frac{F_{m+1,n} - 2F_{m,n} + F_{m-1,n}}{\Delta x^2} \tag{5-39}$$

or

$$F_{m,n+1} = rF_{m-1,n} + (1 - 2r)\,F_{m,n} + rF_{m+1,n} \qquad r = K\,\Delta t/\Delta x^2 \tag{5-40}$$

If, for simplicity, the boundary values are taken to be $F_{0,n} = F_{J,n} = 0$, then (5-40) can be written in the matrix form (5-37).

$$\begin{bmatrix} F_{1,n+1} \\ F_{2,n+1} \\ \cdot \\ \cdot \\ \cdot \\ F_{J-1,n+1} \end{bmatrix} = \begin{bmatrix} 1-2r & r & 0 & 0\ldots & 0 \\ r & 1-2r & r & 0\ldots & 0 \\ 0 & r & 1-2r & r\ldots & 0 \\ & & & & \\ 0 & 0 & \cdots\cdots & \cdots\cdots & 1-2r \end{bmatrix} \begin{bmatrix} F_{1,n} \\ F_{2,n} \\ \cdot \\ \cdot \\ \cdot \\ F_{J-1,n} \end{bmatrix} \tag{5-41}$$

It follows from (5-41) that

$$\mathbf{F}_n = A\,\mathbf{F}_{n-1} = A\,(A\mathbf{F}_{n-2}) = \ldots . A^n\,\mathbf{F}_0 \tag{5-42}$$

where $\mathbf{F}_0$ is the vector representing the initial conditions. The matrix A is referred to as the *amplification* matrix.

It is apparent from (5-42) that the question of computational stability is related to the properties of the matrix A, which will now be discussed as needed. The so-called eigenvalues (or characteristic values) λ_k of A are the roots of the characteristic equation

$$|A - \lambda I| = 0 \tag{5-43}$$

where I is the identity matrix and the bars denote a determinant. In the example (5-41), the determinant is of order $J - 1$, hence the characteristic equation has

the same order and there are $J - 1$ eigenvalues. Associated with each eigenvalue is an eigenvector $\mathbf{v}_k$, which satisfies the equation

$$A\mathbf{v}_k = \lambda_k \mathbf{v}_k \qquad k = 1,2, \ldots \tag{5-44}$$

When the eigenvectors form a complete, linearly independent set, an arbitrary initial condition, such as $\mathbf{F}_0$ in (5-42), can be expressed as a linear combination of the eigenvectors:

$$\mathbf{F}_0 = \sum_k C_k \mathbf{v}_k \tag{5-45}$$

where C_k are constants. A sufficient condition for the matrix A to have a complete set of eigenvectors is that the eigenvalues be nonzero and differ from one another. However, under certain circumstances, a repeated eigenvalue may also lead to a complete set of linearly independent vectors. Also, real symmetric matrices, Hermitian matrices and normal matrices lead to linearly independent sets. In any event, for the examples considered here and usually those of common interest, the sufficiency condition is fulfilled. Assuming this to be the case, substitution of (5-45) into (5-42) and use of (5-44) gives

$$\mathbf{F}_0 = \sum_{k=1} C_k A^n \mathbf{v}_k = \sum_k C_k A^{n-1} A\mathbf{v}_k = \sum_k C_k A^{n-1} \lambda_k \mathbf{v}_k \ldots$$

Repeating this procedure leads to

$$F_n = \sum_{k-1} C_k \lambda_k^n \mathbf{v}_k \tag{5-46}$$

It is evident from (5-46) that the solution $\mathbf{F}_n$ will remain bounded, as required by Definitions I and II for a *stable* difference scheme, if the eigenvalues have magnitudes less or equal to one, that is,

$$|\lambda_k| \leq 1 \text{ for all } k \tag{5-47}$$

In general, a particular nonzero eigenmode will *amplify,* remain *neutral,* or *dampen* according to whether its associated eigenvalue has a magnitude *greater, equal,* or *less than* one,

$$\begin{aligned} &> 1, \text{ amplify} \\ |\lambda_k| &= 1, \text{ neutral} \\ &< 1, \text{ dampen} \end{aligned} \tag{5-48}$$

Strictly speaking, however, the condition (5-47) is more stringent than required for stability according to Definition I. The eigenvalues may be permitted to exceed one by a term having a magnitude no larger than order Δt, that is,

$$|\lambda_k| \leq 1 + 0\,(\Delta t) \tag{5-49}$$

This condition will permit exponential growth of the solution, but the solution remains bounded for a fixed time interval, say T. Such amplification may be fitting and proper for the physical-mathematical system, for example, with baroclinic instability. Further discussion of this case will be given later.

Returning to (5-47), note that a difference system may fail to meet this stability condition by having one (or more) eigenvalues exceed 1 in magnitude, yet if that eigenmode (vector) were not present in the initial state $\mathbf{F}_0$, in theory the solution $\mathbf{F}_n$ would not amplify. However in actual practice, round-off errors occur, hence all modes are soon present and the numerical solution would grow exponentially. A second related point is that if some modes are considered to be undesirable "noise," it may be feasible to filter them from the initial state with the aim of eliminating them from the solution. The elimination may be only temporary, however, since there may be physical or mathematical mechanisms within the difference system, as well as round-off errors, that regenerate the undesirable modes causing them to reappear. If this happens, steps can be taken to suppress them during the integration procedure, perhaps by diffusion terms, or by periodically applying explicit space or time filters. Another possibility is to judiciously choose a differencing scheme that will selectively dampen certain modes.

To consider the matter of error growth further, suppose that errors were to appear at some time, say $t = 0$ for convenience, and assume for simplicity that no further errors are subsequently introduced. If $\mathbf{F}'_0$ represents the contaminated value, the contaminated solution after n time steps, according to (5-42), will be

$$F'_n = A^n \mathbf{F}'_0 \tag{5-50}$$

If the error after n time steps is defined to be

$$\boldsymbol{\epsilon}_n = \mathbf{F}_n - \mathbf{F}'_n$$

it follows from (5-42) to (5-50) that

$$\boldsymbol{\epsilon}_n = A^n \boldsymbol{\epsilon}_0 \tag{5-51}$$

which is precisely the same form as (5-42). Next, $\boldsymbol{\epsilon}_0$ can be expressed in the same manner as (5-45) and $\boldsymbol{\epsilon}_n$ is similar to (5-46):

$$\boldsymbol{\epsilon}_n = \Sigma_k E_k \lambda_k^n \mathbf{v}_k \tag{5-52}$$

Thus the evolution of errors parallels that of the finite difference solution, mode by mode. Since this holds for round-off errors as well as other types, Definition III is a useful practical definition for stability. Although computational stability according to Definitions I and II is unrelated to round-off error, from a pragmatic point of view Definition III is appropriate since actual computation invariably involves round-off error. Moreover, the same condition for the computational

stability of a difference scheme results whether couched in terms of the difference solution or error growth.

Return now to the example used in this section for illustration and express the matrix in (5-41) as the sum of the identity matrix I and a tridiagonal matrix M as follows:

$$A = I + rM$$

where

$$M = \begin{bmatrix} -2 & 1 & 0 & \cdots & 0 \\ 1 & -2 & 1 & \cdots & \\ 0 & 1 & -2 & 1 \cdots & \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots & \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & 1 & -2 \end{bmatrix}$$

It can be shown that if an arbitrary matrix A is a rational function of a second matrix M, then the eigenvalues of A are the same rational function of the eigenvalues of M, that is,

$$\text{if } A = f(M) \qquad \text{then} \qquad \alpha_k = f(\beta_k)$$

where α_k and β_k are the eigenvalues of A and M, respectively. Also the eigenvalues of a tridiagonal matrix of order $J - 1$, with elements a, b, and c in that order, are

$$\beta_k = b + 2\sqrt{ac}\cos\frac{k\pi}{J} \qquad k = 1,2 \ldots J - 1$$

Using the foregoing results gives the following eigenvalues for the matrix A:

$$\lambda_k = 1 - 4r\sin^2\frac{k\pi}{2J} \qquad k = 1,2 \ldots J - 1$$

It is readily seen that

$$|\lambda_k| \leq 1, \text{ provided } r \leq \tfrac{1}{2}, \text{ that is, } K\Delta t/\Delta x^2 \leq \tfrac{1}{2}$$

which is the stability condition for the difference equation (5-39) corresponding to the diffusion equation (5-38).

When the boundary values are not zero or conditions are imposed on the derivatives at the boundaries, the procedure is the same, however the column vectors and the matrix must be augmented. A very important feature of this method is that *boundary conditions can be included* in the analysis. Similarly, if more than two time levels are involved or when there is a system of equations to be solved, the matrix must be augmented accordingly. Some examples will be given later in connection with the von Neumann method of stability analysis.

5-5-2 Von Neumann Method

The von Neumann method is less general than the matrix method just described because the boundary conditions cannot be included, but it is much simpler to apply. It consists of replacing the spatial variation by a single Fourier component. This is sufficient for linear equations with constant coefficients since separate solutions are additive. The method will be illustrated on the *leapfrog* scheme applied to the advection equation (5-6), which must be placed in the form (5-37). However, since there are three time levels in (5-12), it is convenient to introduce a new variable, $G_{m,n} = F_{m,n-1}$. Then (5-12) may be written in the system

$$\begin{aligned} F_{m,n+1} &= G_{m,n} - \frac{c\Delta t}{\Delta x}(F_{m+1,n} - F_{m-1,n}) \\ G_{m,n+1} &= F_{m,n} \end{aligned} \tag{5-53}$$

Next, let

$$F_{m,n} = B_n^{(1)} e^{i\mu m\Delta x} \qquad G_{m,n} = B_n^{(2)} e^{i\mu m\Delta x}$$

Substituting these forms into (5-53) and canceling the common exponential factor leads to a pair of equations that can be put in vector form as follows:

$$\begin{bmatrix} B_{n+1}^{(1)} \\ B_{n+1}^{(2)} \end{bmatrix} = \begin{bmatrix} -2i\sigma & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} B_n^{(1)} \\ B_n^{(2)} \end{bmatrix} \tag{5-54a}$$

or

$$\mathbf{B}_{n+1} = A\mathbf{B}_n = A^2\mathbf{B}_{n-1} \cdots\cdots A^{n+1}\mathbf{B}_0 = \sum_{k=1}^{2} \lambda_k^{n+1} b_k \upsilon_k \tag{5-54b}$$

where the λ_k are the eigenvalues of the 2×2 matrix A; σ is defined by (5-15) and the b_k are constants. The eigenvalues are found by solving the equation

$$\begin{vmatrix} -2i\sigma - \lambda & 1 \\ 1 & -\lambda \end{vmatrix} = 0$$

or

$$\lambda^2 + 2i\sigma\lambda - 1 = 0 \tag{5-55}$$

The coefficients of this equation are identical to those in (5-14), hence the pair of eigenvalues is given by (5-16). As described earlier, a necessary *condition* for *computational stability* is that $|\lambda| \leq 1$ for all eigenvalues. Moreover, from the previous analysis of the roots of (5-55), that is, the right side of (5-16), it is clear that the difference scheme is stable if $|\sigma| \leq 1$, as given by (5-31); and

the guarantee of stability for all admissible wavelengths is $c\Delta t/\Delta x \leq 1$, as given by (5-32). In fact

$$|\lambda_k| \equiv 1 \qquad k = 1,2 \tag{5-56}$$

thus both modes corresponding to the λ's are *neutral*. Note that if the exponential factor is retained when passing from (5-53) to (5-54) the result is similar in form to (5-54) except that the vector **B** is replaced by a vector consisting of components F and G.

The more general criterion for stability given by (5-49), which fulfills the requirements of Definition I—that the solutions of the difference equation remain uniformly bounded functions of the initial state for a finite time T and for sufficiently small Δt and $N\Delta t = T$, will nevertheless, permit exponential growth. To show this, assume

$$\lambda = 1 + a\Delta t \tag{5-57}$$

where a is a constant. Then

$$\lambda^N = (1 + a\Delta t)^N = \left(1 + \frac{aT}{N}\right)^N \qquad N\Delta t = T$$

$$= 1 + N\left(\frac{aT}{N}\right) + \frac{N(N-1)}{2!}\left(\frac{aT}{N}\right)^2 + \cdots\cdots$$

As N increases indefinitely with decreasing Δt, the series approaches

$$\lambda^N = 1 + aT + \frac{(aT)^2}{2!} + \cdots\cdots = e^{aT}$$

Thus, as seen from the various expressions for the solution, (5-20), (5-42), (5-46), or (5-54), those modes associated with eigenvalues of the form (5-57) will display exponential growth. However, within a finite time interval T, the solutions to the difference equation are bounded as $N \to \infty$; and thus the difference scheme is computationally stable in accordance with Definition I.

Even though the differential system may not call for growth, a small amount of growth with a convergent scheme may be practically acceptable, particularly if the scheme has other desirable features, for example, damping of a computational mode. In situations when exponential growth is in accordance with the physics, such as baroclinic or barotropic instability, (5-57) is certainly appropriate.

To pursue this discussion regarding the eigenvalues somewhat further, assume that

$$\lambda = 1 + f(\Delta t) \tag{5-58}$$

where f is an arbitrary function of Δt. Now take the log of (5-58), expand the log in a series and multiply by N giving

$$N \ell n \lambda = N \ell n[1 + f(\Delta t)] \doteq N[f(\Delta t) + \ldots .]$$

Now revert to the exponential form after dropping terms of higher order

$$\lambda^N \doteq e^{N f(\Delta t)}$$

Next consider some examples. First assume $f(\Delta t) = a\Delta t^2$, then

$$\lambda^N = e^{aN\Delta t^2} = e^{aT\Delta t}$$

If $N \to \infty$ and $\Delta t \to 0$, then $\lambda^N \to 1$ and no amplification takes place in the limit, although for a finite Δt some growth will occur for $a > 0$. On the other hand, if $f(\Delta t) = a\Delta t^{1/2}$, then

$$\lambda^N = e^{aN\Delta t^{1/2}} = e^{aT/\Delta t^{1/2}}$$

In this case as $N \to \infty$ and $\Delta t \to 0$, λ^N is unbounded and fails the stability requirement of Definition I.

The von Neumann condition (5-49) on the eigenvalues is certainly necessary for stability of the difference system, but may not be sufficient if the eigenvectors do not form a complete set. Sufficient conditions for a complete set, and thus for stability are somewhat more involved and may be found in standard texts on matrix theory or numerical solutions for partial differential equations, such as Richtmyer and Morton (1967).

5-5-3 The Energy Method

The last method of stability analysis to be mentioned here is the so-called *energy* method, which may or may not have anything to do with physical forms of energy. This method provides a sufficient condition for stability and is also applicable to nonlinear equations. If the true solution is known to be bounded, then the finite difference solution should be examined for boundedness. If it can be determined that the sum $\sum_m (u_{m,n})^2$, over all gridpoints for any time n, is bounded, then every $u_{m,n}$ is also bounded and stability is established. This method is usually considerably more difficult to apply than the von Neumann method, but it sometimes is usable on systems where the von Neumann method is inapplicable. In this text, except for a simple example, we confine the discussion henceforth to the von Neumann method, which is applicable to a wide variety of linear problems and serves as a useful guide to nonlinear problems.

5-6 EXAMPLES OF THE VON NEUMANN METHOD

In this section further examples of stability analysis will be given including iterative schemes, two space dimensions, multiple equations, etc.

5-6-1 Euler Scheme

First consider the two-level forward difference scheme called the *Euler scheme,* which was used to determine the computational boundary condition (5-22) for the leapfrog scheme,

$$F_{m,n+1} = F_{m,n} - \frac{c\Delta t}{2\Delta x}(F_{m+1,n} - F_{m-1,n}) \tag{5-59}$$

It is readily shown that the truncation error is of order, $O(\Delta t) + O(\Delta x^2)$ (student exercise). Assuming a solution of the form

$$F_{m,n} = B^{n\Delta t}\, e^{i\mu m\Delta x} \tag{5-60}$$

and substituting into (5-59) gives

$$B^{\Delta t} = 1 - \sigma i = (1 + \sigma^2)^{1/2}\, e^{-i\theta}$$

where σ is given by (5-15) and $\theta = \arctan \sigma$. After utilizing the initial condition, (5-60) becomes

$$F_{m,n} = A\left[1 + \left(\frac{c\Delta t}{\Delta x}\right)^2 \sin^2 \mu\Delta x\right]^{n/2} e^{i\mu(m\Delta x - n\theta/\mu)}$$

It is immediately apparent that the Euler scheme gives exponential growth for almost all wavelengths.[2] Consequently, it is of little consolation to note that no computational mode in time appears, the reason being that the difference equation is first order in time as is the differential equation.

In order to facilitate other stability analyses, it is convenient to note that for a solution of the form (5-60), the centered difference may be expressed in the form

$$F_{m+1,n} - F_{m-1,n} = (2i \sin \mu\Delta x)\, F_{m,n} \tag{5-61}$$

5-6-2 Uncentered Differencing, Von Neumann Method

As a slight variation from the Euler scheme, assume the forward time difference but a backward space difference to approximate the advection equation, that is,

$$F_{m,n+1} = F_{m,n} - (c\Delta t/\Delta x)(F_{m,n} - F_{m-1,n}) \tag{5-62}$$

With $c > 0$, the space derivative may be described as "upstream," and for $c < 0$, "downstream," especially when $c = U$ is the mean wind velocity. Assuming the usual solution form, (5-13) gives

[2] Strictly speaking if one were able to fix μ, then the scheme would be stable and convergent; however, in practice this is not feasible for reasons of round-off error alone, which can introduce arbitrarily large wave numbers.

$$F_{m,n+1} = [1 - \sigma(1 - e^{-i\mu\Delta x})]\, F_{m,n} \qquad \sigma = c\Delta t/\Delta x$$

The eigenvalue for this single element matrix is just the element

$$\lambda = 1 - \sigma + \sigma \cos \mu\Delta x - \sigma i \sin \mu\Delta x$$

and

$$|\lambda|^2 = (1 - \sigma + \sigma \cos \mu\Delta x)^2 + \sigma^2 \sin^2 \mu\Delta x$$

$$= 1 - 2\sigma(1 - \cos \mu\Delta x)(1 - \sigma), \text{ a parabola in } \sigma$$

Note that when:

$\sigma = 0$ or $\sigma = 1$,	$\lvert\lambda\rvert = 1$	neutral
$\sigma < 0$ or $\sigma > 1$,	$\lvert\lambda\rvert > 1$	amplified
$0 < \sigma < 1$,	$\lvert\lambda\rvert < 1$	damped

Thus with *upstream* differencing, $c > 0$, *damped or neutral solutions* are the norm, provided $0 \leq c\Delta t/\Delta x \leq 1$. On the other hand, with *downstream* differencing ($c < 0$), $\sigma < 0$, and the solutions are *amplified* (*computationally unstable*). (For stability with $c < 0$, the space differencing would involve points m and $m + 1$.)

If F is taken as temperature and c, the wind velocity, it would be natural to look upstream when forecasting temperature changes by advection, rather than downstream. The numerical solution bears this out, although there is erroneous damping (Figure 5-3*b*). Upstream differencing is sometimes used for moisture advection, despite the damping, because negative values of moisture will not be produced, as may happen with central differencing (Figure 5-3).

Energy Method. The upstream case is a particularly easy one to illustrate the energy method which is also applicable to nonlinear equations. For this purpose F is assumed to be cyclic so that $F_{-1,n} = F_{M,n}$, and $x = m\Delta x$, $m = 0,1, \ldots M$. The solution is assumed to be bounded initially and it will be shown that the following relationship holds (with the proper restriction on σ):

$$\sum_m F^2_{m,n+1} \leq \sum_m F^2_{m,n} \tag{5-63}$$

This condition will ensure that the sum of the squares of $F_{m,n}$ does not increase with n. Moreover, it is evident that the individual $F_{m,n}$ must be also bounded if the sum of the squares of all $F_{m,n}$ is bounded. Thus (5-63) is a *sufficient* condition for *computational stability*.

First square both sides of (5-62), collect terms and then sum over m giving

$$\sum F^2_{m,n+1} = \sum_m [(1 - \sigma)^2 F^2_{m,n}$$

$$+ 2\sigma(1 - \sigma) F_{m,n} F_{m-1,n} + \sigma^2 F^2_{m-1,n}] \tag{5-64}$$

Two relationships will be derived for use in (5-64). The first follows directly from the cyclic continuity:

$$\sum_{m=0}^{M} F_{m-1,n}^2 = \sum_{m=0}^{M} F_{m,n}^2$$

The second relationship derives from the foregoing result and the simple fact that the square of a real quantity is positive as follows:

$$(F_{m,n} - F_{m-1,n})^2 = F_{m,n}^2 - 2F_{m,n}F_{m-1,n} + F_{m-1,n}^2 \geq 0$$

a b c

Transposing the middle term (b) summing, then combining the first and last sums (a and c) gives

$$\sum_m F_{m,n}^2 \geq \sum_m F_{m,n} F_{m-1,n}$$

Now return to (5-64) and make use of these two relationships successively to give

$$\begin{aligned} \sum F_{m,n+1}^2 &= \sum_m \{[(1-\sigma)^2 + \sigma^2] F_{m,n}^2 + 2\sigma(1-\sigma) F_{m,n} F_{m-1,n}\} \\ &\leq \sum_m [(1-\sigma)^2 + \sigma^2 + 2\sigma(1-\sigma)] F_{m,n}^2 = \sum_m F_{m,n}^2 \end{aligned} \quad (5\text{-}65)$$

The substitution leading to the inequality is, of course, valid only if the coefficient $2\sigma(1-\sigma)$ is positive. The latter follows if $0 < \sigma \leq 1$, which is precisely the condition for computational stability of upstream differencing derived earlier in this section. The last equality holds because the coefficient of $F_{m,n}^2$ is exactly 1. Thus (5-63) has been established. The proof of (5-63) is considerably more complicated for most differencing schemes.

5-6-3 Trapezoidal Implicit Scheme

Both the forward and central time differencing schemes discussed previously have problems associated with them in addition to truncation error, the Euler scheme being unstable and the leapfrog, though conditionally stable, having a computational mode. The next technique, a trapezoidal scheme, avoids the two foregoing problems but, as will be seen, has another complication. This procedure uses only two time levels and thus avoids the computational mode, while using central differences for the space derivative centered at $[n + (1/2)]\Delta t$ by averaging values at n and $n + 1$ as follows

$$\frac{F_{m,n+1} - F_{m,n}}{\Delta t} = -\frac{c}{2}\left(\frac{F_{m+1,n+1} - F_{m-1,n+1}}{2\Delta x} + \frac{F_{m+1,n} - F_{m-1,n}}{2\Delta x}\right) \quad (5\text{-}66)$$

Using (5-61) on the central differences easily leads to

$$F_{m,n+1} = \left(\frac{1 - i(c\Delta t/2\Delta x)\sin \mu\Delta x}{1 + i(c\Delta t/2\Delta x)\sin \mu\Delta x}\right) F_{m,n} \equiv \lambda F_{m,n}$$

The eigenvalue λ is again just the coefficient of $F_{m,n}$, the amplification matrix. Since the magnitude of the ratio of two complex numbers is just the ratio of their magnitudes, it follows immediately that

$$|\lambda| \equiv 1, \textit{ unconditionally stable} \text{ for all } \Delta x \text{ and } \Delta t$$

By expanding in series about the time $n + \frac{1}{2}$, it may be shown that the implicit scheme has a truncation error of second order in t and x [i.e., $O(\Delta t)^2 + O(\Delta x)^2$].

The complication referred to earlier is that (5-66) no longer provides a simple marching process. Suppose the calculations have been completed for all gridpoints m at time n and it is desired to proceed to time $n + 1$. Then (5-66) may be solved for $F_{m,n+1}$, but the quantity, $F_{m+1,n+1}$, which is not yet known, appears on the right side. This means that a system of simultaneous equations for the F's must be solved, rather than simply progressing one at a time as in the leapfrog scheme (5-12). The latter is referred to as an *explicit* scheme and (5-66), an *implicit* scheme. The number of simultaneous equations will essentially equal the number of gridpoints, perhaps many thousands, and this involves inverting, in one form or another, a large matrix, which is very time consuming even with a high-speed computer. Consequently, implicit methods have been little used in meteorology; however some semiimplicit techniques are feasible.

The equation following (5-66) may be written

$$F_{m,n} = \lambda F_{m,n-1} = \ldots = \lambda^n F_0 \equiv A\lambda^n e^{i\mu m\Delta x}$$

$$= Ae^{i\mu(m\Delta x - 2n\theta/\mu)}$$

where

$$\theta = \arctan\left[(c\Delta t/2\Delta x)\sin \mu\Delta x\right]$$

It is readily seen that

$$c_F = 2\theta/\Delta t\mu$$

Now when $L = 2\Delta x$, $\mu\Delta x = \pi$ and $\theta = 0$; hence $c_F = 0$. For large L, $\mu\Delta x << 1$ and $\sin \mu\Delta x \doteq \mu\Delta x$. Thus $\theta \doteq \arctan(c\mu\Delta t/2)$. Continuing, for small $c\mu\Delta t/2$, $\theta \doteq c\mu\Delta t/2$ and $c_F \doteq c$; whereas for large $c\mu\Delta t/2$, $\theta \doteq \pi/2$; and $c_F = \pi/\Delta t\mu$, which does not depend on c. It may be inferred that for large Δt, which will not create instability with this scheme, c_F will be in serious error. Thus the *implicit scheme,* although *unconditionally stable,* has serious errors in the phase velocity not only for short waves as in the leapfrog scheme, but also for relatively large $c\Delta t/\Delta x$. The implications will be discussed later in