Solution of homogeneous and nonhomogeneous 1d diffusion equations on the surface of a sphere, as function of latitude

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Contents

1 Equation

Diffusion equation on the surface of a sphere,

$$\frac{\partial u}{\partial t} = \hat{D}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right)\frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial\phi^2}\right]$$

for a temperature which is a function of co-latitude and time only, $u = u(\theta, t)$,

$$\frac{\partial u}{\partial t} = \hat{D}\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right)$$

Defining $x = \cos\theta$ we have (see notes in power method chapter)

$$dx = \sin\theta d\theta,$$
$$\frac{d}{d\theta} = \sin\theta\frac{d}{dx}$$

so that, defining $D = \hat{D}/r^2$,

$$\frac{\partial u}{\partial t} = D\frac{\partial}{\partial x}\left((1-x^2)\frac{\partial u}{\partial x}\right)$$

2 Homogeneous case

Letting $u = T(t)X(x)$, we have

$$\frac{1}{D}\frac{T'}{T} = \frac{((1-x^2)X')'}{X} = -\kappa^2$$

so that,

$$T' = -D\kappa^2 T$$
$$((1-x^2)X')' + \kappa^2 X = 0.$$

Solution which is bounded for $x = \pm 1$ is $\kappa^2 = n(n+1)$, for $n = 0, 1, 2, \ldots$ and

$$T(t) = Ae^{-D\kappa^2 t}$$
$$X(x) = CP_n(x)$$

where $P_n(x)$ are Legendre polynomials. So general solution which satisfies the boundary conditions is

$$u(x,t) = \sum_{n=0}^{\infty} C_n P_n(x) e^{-Dn(n+1)t}$$

impose initial conditions,

$$u(x, t=0) = f(x) = \sum_{n=0}^{\infty} C_n P_n(x).$$

To calculate C_n, multiply both sides by $P_m(x)$ and integrate over x, using the orthogonality condition for Legendre polynomials, (weight function is 1, see Greenberg p 910, section 17.8),

$$\int_{-1}^{1} P_n(x)P_m(x)dx = \delta_{mn}\frac{2}{2m+1},$$

to find,

$$C_m = \frac{2m+1}{2}\int_{-1}^{1} f(x)P_m(x)dx,$$

which concludes the solution, see demo Matlab program `diffusion_1d_sphere_SL.m`.

3 Nonhomogeneous case

Consider temperature as function of latitude, $u(x)$, forced by solar radiation as function of latitude, $S(x)$,

$$\frac{du}{dt} = D\frac{d}{dx}\left((1-x^2)\frac{du}{dx}\right) + S(x).$$

Expand the forcing in Legendre polynomials,

$$S(x) = \sum_{n=0}^{\infty} S_n P_n(x).$$

where

$$S_n = \frac{\int_{-1}^{1} S(x)P_n(x)dx}{\int_{-1}^{1} P_n(x)P_n(x)dx} = \frac{2n+1}{2}\int_{-1}^{1} S(x)P_n(x)dx,$$

and divide the solution into particular solution which is time independent (steady state), and time dependent,

$$u(x,t) = \overline{u}(x) + \hat{u}(x,t).$$

Steady part satisfies,

$$0 = D\frac{d}{dx}\left((1-x^2)\frac{d\overline{u}(x)}{dx}\right) + S(x).$$

so that expanding

$$\overline{u}(x) = \sum_{n=0}^{\infty} \overline{U}_n P_n(x)$$

and substituting this and the expansions for $S(x)$ into the steady equation using also,

$$\frac{d}{dx}\left((1-x^2)\frac{dP_n(x)}{dx}\right) + n(n+1)P_n(x) = 0,$$

we find

$$0 = -D\sum_{n=0}^{\infty} \overline{U}_n n(n+1)P_n(x) + S_n P_n(x),$$

from which we find the coefficients for the steady state part,

$$\overline{U}_n = \frac{S_n}{Dn(n+1)}.$$

Note that $S_0 = 0$ (so that $U_0 = 0$ as well), for a steady state to exist (it can only exist if the net forcing vanishes, and S_0 is proportional to the spatial mean of the forcing, because $P_0(x) = constant$).

The remaining, time dependent, part of the solution satisfies

$$\frac{d\hat{u}}{dt} = D\frac{d}{dx}\left((1-x^2)\frac{d\hat{u}}{dx}\right).$$

with initial conditions,

$$\hat{u}(x,t=0) = u(x,t=0) - \overline{u}(x),$$

which is a homogeneous problem equivalent to that we already solved in the previous section.

Note that for a steady solution to exist, consistency conditions on the integral of $S(x)$ must be satisfied, something like vanishing integral over x.