Similarity solutions of the Diffusion Equation

The diffusion equation in one-dimension is

$$u_t = \kappa u_{xx} \tag{1}$$

where κ is the diffusion coefficient which has dimension L^2T^{-1}. If we are looking for solutions of (1) on an infinite domain $-\infty \le x \le \infty$ where there is no natural length scale, then we can use the dimensionless variable

$$\eta = \frac{x}{\sqrt{\kappa t}} \tag{2}$$

and look for solutions of (1) in the form

$$u(x,t) = t^p g(\eta) \tag{3}$$

where the number p and the function $g(\eta)$ are to be determined. Substituting (3) into (1) we find that

$$t^{p-1}\left(pg - \frac{\eta}{2}g' - g''\right) = 0 \tag{4}$$

and so

$$g'' + \frac{\eta}{2}g' = pg. \tag{5}$$

This is difficult to solve for arbitrary values of p but for special values we can do something.

1. Take $p = 0$ and (5) is easily solved to give

$$g'(\eta) = A\,e^{-\eta^2/4} \tag{6}$$

   where A is a constant. Integrating again we have

$$g(\eta) = A\int_{-\infty}^{\eta} e^{-\eta'^2/4}\,d\eta'. \tag{7}$$

   This gives a full solution for $u(x,t)$

$$u(x,t) = A\int_{-\infty}^{\frac{x}{\sqrt{\kappa t}}} e^{-\eta'^2/4}\,d\eta' = 2A\sqrt{\pi}\,\mathrm{erf}\left(\frac{x}{2\sqrt{\kappa t}}\right) \tag{8}$$

   where the *error function* $\mathrm{erf}\,(\xi)$ is defined as $\mathrm{erf}\,(\xi) = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\xi} e^{-y^2}\,dy$. This has the property that $\mathrm{erf}\,(\infty) = 1$.

2. Now define $G = g\,e^{\eta^2/4}$ and we observe that 5) can be transformed into

$$G'' - \frac{\eta}{2}G' = (p + 1/2)G. \tag{9}$$

   This has the trivial solution $G = b = \text{const}$ provided $p = -1/2$. Hence

$$g(\eta) = b\,e^{-\eta^2/4}. \tag{10}$$

   This gives a full solution for $u(x,t)$ in the form

$$u(x,t) = b\,t^{-1/2}e^{-\frac{x^2}{4\kappa t}}. \tag{11}$$