Topics:

1. Phase plane

2. Index Theory

3. Liapunov Functions

   (a) Have the following properties:
      i. \( V(x) > 0 \) for all \( x \neq x^* \), \( V(x^*) = 0 \)
      ii. \( V < 0 \) for all \( x \neq x^* \)

   (b) If we can find one, there are no closed orbits.

   (c) \( V(x) = x^2 + ay^2 \) is a common guess.

4. Gradient systems \((\dot{x} = -\nabla V)\) have no closed orbits.

Examples:

1. Show that \( \dot{x} = -x + 2y^3 - 2y^4 \), \( \dot{y} = -x - y + xy \) has no periodic solutions. *Hint:* use the Liapunov function \( V = x^m + ay^n \).
2. Consider the system $\dot{x} = x - x^2$.

   (a) Find the fixed points and classify them.

   (b) Find the conserved quantity (the “energy”).
(c) Sketch the phase portrait.

(d) Find an equation for the homoclinic orbit that separates closed and non-closed trajectories.
3. A two-dimensional dynamical system has a circle in the phase plane on which the flows are inwards. It is known that inside the circle there is an unstable clockwise spiral fixed point and a stable counterclockwise limit cycle that does not enclose the fixed point, as shown below.

(a) Use index theory to explain why there must be at least two other fixed points inside the circle. If there are exactly two, what type of fixed points can they be?

(b) On the figure, indicate possible locations for the two extra fixed points and sketch a plausible phase portrait.