Topics:

1. Taylor Series (special request from Laure and Professor Tziperman). The function \( f(x,y) \), with continuous second-order partial derivatives, can be expanded near \((a,b)\) as:

\[
f(x,y) \approx f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) + \frac{f_{xx}(a,b)}{2}(x-a)^2 + f_{xy}(a,b)(x-a)(y-b) + \frac{f_{yy}(a,b)}{2}(y-b)^2
\]

Here subscripts denote partial derivatives.

2. The circle map, defined as:

\[
F(\theta) = \theta + \Omega - \frac{K}{2\pi} \sin(2\pi \theta) \mod 1
\]

Or without the modulo:

\[
\tilde{F}(\theta) = \theta + \Omega - \frac{K}{2\pi} \sin(2\pi \theta)
\]

The second form is used for the winding number.

3. Fixed points in linear and nonlinear 2-D systems. The equation to remember: \( \lambda_{1,2} = \frac{1}{2}(\tau \pm \sqrt{\tau^2 - 4\Delta}) \). \( \tau \) and \( \Delta \) are the trace and determinant of the Jacobian evaluated at the fixed point. We use the eigenvalues to classify the fixed point.
Examples:

1. The simplest model in which two species, with populations $x$ and $y$, compete for the same resources has the form: $\dot{x} = rx(1-ax-by)$, $\dot{y} = qy(1-fy-gx)$ with all the parameters positive. This system can be rescaled: $\frac{dX}{d\tau} = X(1-X-AY)$, $\frac{dY}{d\tau} = BY(1-Y-CX)$ with $\tau = rt$, $X=ax$, $Y=fy$, $A=b/f$, $B=q/r$, $C=g/a$. $A$ and $C$ are measures of the inter- and intra- species competition. For example, if $A>1$ increasing $Y$ lowers the growth rate of $X$ more than it lowers the growth rate of $Y$.

(a) Find and classify the fixed points.
(b) Show that there are four qualitatively different phase portraits, find the conditions on the parameters under which each of them obtains, and sketch an example of each.

(c) Can we get stable coexistence of the two species for any of the possibilities? What does the relationship between inter- and intra- species competition need to be for this to occur?
2. Consider the system \( \dot{x} = -y - x^3, \dot{y} = x \) whose linearization suggests that the origin is a center. By considering the behavior of \( r^2 = x^2 + y^2 \) show that the trajectories actually spiral toward the origin. A nonlinear system must be reversible (invariant under \( t \rightarrow -t, y \rightarrow -y \)) for nonlinear centers to exist.