Half Stable Fixed Points on Maps

There was a question in section today about whether you can have half stable fixed points with maps. This would require \( x^* = f(x^*) \) (so that \( x^* \) is a fixed point), \( \frac{df}{dx} \) crossing 1 at \( x = x^* \) (so that the stability switches at \( x = x^* \)). Here is an example of a map with such properties:

\[
x_{n+1} = f(x_n) = \begin{cases} 
1 + \ln(x_n) & x > 0 \\
0 & x \leq 0
\end{cases}
\]

The definition for \( x \leq 0 \) is necessary to avoid taking the log of a non-positive number, but isn’t important to us. The fixed point is \( x^* = 1 \) and \( \frac{df}{dx} \bigg|_{x=1} = 1/x \bigg|_{x=1} = 1 \). \( \frac{df(x<1)}{dx} > 1 \) and \( \frac{df(x>1)}{dx} < 1 \) so the fixed point \( x^* = 1 \) is unstable on the left side and stable on the right side - it is half-stable.

The following figure shows this: