Problem Set 12 Solutions

1. The time series is plotted for $\dot{q}$ instead of $q$ itself because $q$ may become very large if the pendulum winds around the top and it may be hard to understand the system’s behavior.

   (a) For $k=0.5$ we see similar behavior to the transition to chaos in the circle map. We can enter and leave mode-locked regions (Arnold tongues) and have quasi-periodic behavior between them. After the onset of chaos we can still get mode-locked regions.

![Figure 1: $f=0.1$, a mode-locked solution (periodic).](image1)

![Figure 2: $f=0.5$, a quasi-periodic solution. The bagel in phase space and circle in the Poincare section signify this.](image2)
Figure 3: $f=1.25$, we enter another mode-locked solution. Interestingly, from here we follow a period-doubling route to chaos.

Figure 4: $f=1.3$, the period has doubled.
Figure 5: $f=1.31$, the period has doubled again.

Figure 6: $f=1.4$, chaos! If we keep increasing $f$ we will reach a phase-locked solution near $f=1.9$ that will go through another period-doubling cascade back to chaos.
(b) When $k=0$, we don’t find any quasi-periodic behavior. So we might consider this a degenerate case of the quasi-periodic route to chaos. The quasi-periodicity is lost due to the symmetry gained by setting $k=0$.

Figure 7: For small forcing ($f=0$) the system is damped.

Figure 8: At larger forcing ($f=0.5$) we get mode-locked behavior.
Figure 9: Then the system becomes chaotic ($f=1$).

Figure 10: You can find mode-locked periodic behavior after the onset of chaos ($f=1.1$ here).
(c) I was able to get a very rough picture of what was going on using this method. A major annoyance is that the behavior seems to depend pretty strongly on the time-stepping. When I decreased the time-stepping by a factor of 10 the behavior changed dramatically. To give you an idea of how complex an actual state plot of a system like this is, here’s one from (D’Humieres et al., 1982) for k=0.

Figure 11: The x-axis is the frequency of the driving and the y-axis is the coefficient of the periodic forcing.

2. In general when the map of an experimental system that is similar to the circle map becomes non-invertible, chaos occurs.

Figure 12: For K<1 $\theta_{n+1}$ is a single-valued function of $\theta_n$ and vice versa, so the circle map is invertible for K<1.
Figure 13: For $K>1$ $\theta_{n+1}$ is a single-valued function of $\theta_n$, but $\theta_n$ is not a single-valued function of $\theta_{n+1}$, so the circle map is not invertible for $K>1$. I had to use 30 random starting points to get a decent picture of the function here.
3. The nth iteration of the Koch curve has $4^n$ intervals of length $(\frac{1}{3})^n$. So the box dimension is:

$$D_{\text{box}} = \lim_{\varepsilon \to 0} \frac{\ln(N(\varepsilon))}{\ln(\frac{1}{\varepsilon})}$$

$$= \lim_{n \to \infty} \frac{\ln(4^n)}{\ln(3^n)}$$

$$= \frac{\ln(4)}{\ln(3)}$$

$$\approx 1.26$$  \hspace{1cm} (1)

4. The nth iteration has $3^{2n}$ 2D boxes each with side length $\varepsilon = (\frac{1}{3})^n$. $4^n$ are filled. The box dimension is:

$$D_{\text{box}} = \lim_{\varepsilon \to 0} \frac{\ln(N(\varepsilon))}{\ln(\frac{1}{\varepsilon})}$$

$$= \lim_{n \to \infty} \frac{\ln(4^n)}{\ln(3^n)}$$

$$= \frac{\ln(4)}{\ln(3)}$$

$$\approx 1.26$$  \hspace{1cm} (2)

The same as the Koch curve!

5. We have a 3D cube, such that divide into 27, we obtain $27 = 3^3$ cubes. After one iteration, we removed $2 \cdot 3 + 1$ cubes (2 · 3 faces and 1 center). We are left with $[3^3 - (2 \cdot 3 + 1)]$ cubes. After 2 iterations $[3^3 - (2 \cdot 3 + 1)]^2$… After n iterations, $[3^3 - (2 \cdot 3 + 1)]^n$ each with side length $\varepsilon = (\frac{1}{3})^n$. The box dimension is:

$$D_{\text{box}} = \lim_{\varepsilon \to 0} \frac{\ln(N(\varepsilon))}{\ln(\frac{1}{\varepsilon})}$$

$$= \lim_{n \to \infty} \frac{\ln[3^3 - (2 \cdot 3 + 1)]^n}{\ln(3^n)}$$

$$= \frac{\ln(20)}{\ln(3)}$$

$$\approx 2.73$$  \hspace{1cm} (3)

6. Now imagine we have an N-dimensional cube and we keep the only the corners at each fractal iteration. We still have $\varepsilon = (\frac{1}{3})^n$ for the nth iteration. So the box dimension is:

$$D_{\text{box}} = \lim_{\varepsilon \to 0} \frac{\ln(N(\varepsilon))}{\ln(\frac{1}{\varepsilon})}$$

$$= \lim_{n \to \infty} \frac{\ln[3^N - (2 \cdot N + 1)]^n}{\ln(3^n)}$$

$$= \lim_{n \to \infty} \frac{\ln[3^N - (2 \cdot N + 1)]}{\ln(3)}$$  \hspace{1cm} (4)
7. There are $4^{2n}$ 2D boxes at the nth iteration, so $\varepsilon = \left(\frac{1}{4}\right)^n$. 8 of these boxes are filled. So the box dimension is:

$$D_{\text{box}} = \lim_{\varepsilon \to 0} \frac{\ln(N(\varepsilon))}{\ln(\frac{1}{\varepsilon})}$$

$$= \lim_{n \to \infty} \frac{\ln(8^n)}{\ln(4^n)}$$

$$= \frac{\ln(8)}{\ln(4)}$$

$$= \frac{3}{2} \quad \text{(5)}$$