Homework #8
Nonlinear dynamics and chaos

1. **Global bifurcation of limit cycles:** (Strogatz 8.4.2) Analyze the bifurcations as function of $\mu$ of the system

   \[
   \begin{align*}
   \dot{r} &= r(\mu - \sin r) \\
   \dot{\theta} &= 1 
   \end{align*}
   \]

2. **Hopf bifurcation in Lorenz equations:** Find the critical $r_H$ at which a Hopf bifurcation of the $C^+, C^-$ points occurs in the Lorenz system.

3. **Pitchfork bifurcation in the Lorenz equations:** Plot the fixed points for $x, y$ and $z$ as function of $r$ for the Lorenz system. Plot also $x^2 + y^2 + z^2$ for these fixed points as function of $r$. Explain each of your plots. Note that this is a 3d system that undergoes the pitchfork bifurcation whose normal form is 1d.

4. **Hysteresis for the driven pendulum:** (Numerical, use driven_pendulum.m from the course home page).

   (a) Find values of the friction $\alpha$ and forcing $I$ in the equation $\phi'' + \alpha \phi + \sin \phi = I$ for which there are both a stable limit cycle and a stable fixed point. Solve numerically, show and explain how the system approaches these two different solutions for different initial conditions.

   (b) Estimate the period as function of the bifurcation parameter $I$ for $\alpha = 1.5$ as $I$ approaches 1 from above. Plot $\text{period}(I)$ together with the expected dependency for this kind of a bifurcation. Discuss the results. Plot the oscillations for $I = 1.001$ or for similar value just above 1. This form of oscillations is typically found in experimental or model systems for infinite period bifurcations.

5. **Numerical integration of the Lorentz system:** Set $b = 8/3; \sigma = 10$. Use the solver lorenz.m on the course home page to plot the time series of the Lorenz system in the regimes (a) $r < 1$; (b) $1 < r < r_H$; (c) $r = r_h + \varepsilon$ for some small $\varepsilon$, (in this case, start with initial conditions very close to the location of one of the $C^+/C^-$ fixed point; (d) $r = 28$. For each of these values of $r$, plot a time series of $y(t)$ as well as a phase trajectory in the $(x,z)$ plane, and explain what you see in terms of the bifurcation behavior of $r$ analyzed in class.