1. **Circle map experimentation and Phase locking:** Consider the circle map,

\[
\theta_{n+1} = \theta_n + \Omega - \frac{K}{2\pi} \sin(2\pi\theta_n) \mod 1 \tag{1}
\]

a Matlab program for the circle map is on the course home page.

(a) Find a period 2 solution \((p/q = 1/2)\) for \(K > 1/2, \Omega \neq 1/2\).

i. Show that the solution does not depend on the initial conditions \(\theta_0\) by iterating the map from 2 different initial conditions, converging to the same period-2 solution.

ii. Find another \((p/q = 1/2)\) solution for different \(K, \Omega\), with again \(K > 1/2, \Omega \neq 1/2\), show that it is different from the previous one although it has the same period

(b) Find a \(p/q = 0/1\) solution for \(\Omega \neq 0, K > 1/2\). Describe the behavior of this solution as function of \(n\). Do the same for a \(p/q = 1/1\) solution for \(\Omega \neq 1, K > 1/2\).

(c) Find a solution \(p/q = 3/4\) for \(K > 1/2\).

(d) Try a few cases with \(K > 1\). What happens?

2. **Linearized 2d systems:**

   Classify the stability of the fixed points of the following systems by solving for their eigenvalues/ vectors and plotting the vector field in the phase plane. If the eigenvectors are real, plot them in phase space. Can use the quiver function of Matlab for the plot.

   \[
   \begin{align*}
   \dot{x} &= x - y; & \dot{y} &= x + y & (2) \\
   \dot{x} &= 5x + 2y; & \dot{y} &= -17x - 5y & (3) \\
   \dot{x} &= 5x + 10y; & \dot{y} &= -x - y & (4)
   \end{align*}
   \]

3. **Nonlinear 2d systems:**

   Find the fixed points, classify them, sketch neighboring trajectories, and try to fill in the rest of the phase space portrait:

   \[
   \begin{align*}
   \dot{x} &= x - y; & \dot{y} &= x^2 - 4 & (5)
   \end{align*}
   \]
\[
\begin{align*}
\dot{x} &= \sin y; \quad \dot{y} = \cos x \quad (6) \\
\dot{x} &= xy - 1; \quad \dot{y} = x - y^3 \quad (7) \\
\dot{x} &= xy; \quad \dot{y} = x^2 - y \quad \text{beware, linearization fails here. why?} (8)
\end{align*}
\]

4. **Analytic determination of Arnold tongues in circle map near** \(K = 0\): Consider the circle map

\[
\theta_{n+1} = F(\theta_n) = \theta_n + \Omega - \frac{K}{2\pi} \sin(2\pi \theta_n) \quad \text{(mod 1)}. \quad (9)
\]

It can be shown that the rotation (winding) number, defined as the limit

\[
w = \lim_{n \to \infty} (\theta_n - \theta_o)/n \quad \text{(where \(\theta_n\) is not taken mod 1 for the purpose of calculating the winding number)}
\]

is \(p/q\) if and only if

\[
F^q(\theta) - (\theta + p) = 0. \quad (10)
\]

First, test this numerically for some two different values of \((p,q)\). **(Optional yet easy):** prove that if condition (10) is satisfied, the winding number is indeed \(p/q\). Next, use this relation to show that the edges of the Arnold tongues \(q = 1\) and \(p = 0\) or \(p = 1\) are at \(\Omega = K/(2\pi)\) (for \(p = 0\)) and \(\Omega = 1 - K/(2\pi)\) (for \(p = 1\). Hint: use (9) and (10) to find the condition satisfied by \(\theta_n\) within the appropriate tongue, and then deduce a condition for the edge of the tongue.

5. **Challenge/ Extra credit:** Using the same approach as the previous question, show that the boundary of the Arnold tongues \(p = 1\) and \(q = 2\) for small \(K\) is given by \(\Omega = \frac{1}{2} \pm \frac{K^2}{8\pi}\). Hint: write \(\Omega = 1/2 + \varepsilon\) and expand the relevant equations in terms of the small parameters \(K\) and \(\varepsilon\).