Problem Set 4 Solutions

1) a) Here the circle map is integrated at two different K, Ω combinations that both produce p/q=1/2 for three initial conditions each.
b) The p/q=0/1 plot is above and the p/q=1/1 plot is below. They look similar - each settles into a period one pattern after some initial behavior.
d) For $K>1$ chaotic and nonchaotic regions are densely interwoven in the $K-\Omega$ plane.
2) a) The f.p. is at \((x,y) = (0,0)\). \(A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, \lambda_1 = 1 + i, v_1 = \begin{bmatrix} 1 \\ -i \end{bmatrix}, \lambda_2 = 1 - i, v_2 = \begin{bmatrix} 1 \\ i \end{bmatrix}\). An unstable spiral.
b) The f.p. is at (x,y)=(0,0). $A=[5 \ 2; -17 \ -5]$, $\lambda_1=3i$, $v_1=[2 \ -5+3i]$, $\lambda_2=-3i$, $v_2=[2 \ -5-3i]$. A center.
c) The f.p. is at \((x,y)=(0,0)\). \(A=[5 \ 10; -1 \ -1]\), \(\lambda_1=2+i, \ v_1=[10 \ -3+i]\), \(\lambda_2=2-i, \ v_2=[10 \ -3-i]\). An unstable spiral.

\[
\begin{align*}
x' &= 5x + 10y \\
y' &= -x - y
\end{align*}
\]
3) a) The f.p. are at \((x,y)=(2,2)\) and \((-2,-2)\). The Jacobian is \(J=[1 \ -1; \ 2x \ 0]\).

\[
A_1 = J|_{(-2,-2)} = \begin{bmatrix} 1 & -1; -4 & 0 \end{bmatrix} \Rightarrow \tau = 1, \ \Delta = -4 \Rightarrow \lambda_1 = \frac{1}{2} + \sqrt{\frac{17}{2}}, \ v_1 = [2; 1 - \sqrt{17}], \ \lambda_2 = \frac{1}{2} - \sqrt{\frac{17}{2}}, \ v_2 = [2; 1 + \sqrt{17}]\]. A saddle point.

\[
A_2 = J|_{(2,2)} = [1 \ -1; 4 \ 0] \Rightarrow \tau = 1, \ \Delta = 4 \Rightarrow \lambda_{1,2} = \frac{1}{2} \pm \frac{\sqrt{17}}{2}. \ An \ unstable \ spiral.
\]
b) The f.p. are at \((x,y) = ((n + \frac{1}{2})\pi, m\pi)\) for integer \(n\) and \(m\). The Jacobian is \(J = \begin{bmatrix} 0 & \cos(y) \\ -\sin(y) & 0 \end{bmatrix}\).

\[ A = J_{((n + \frac{1}{2})\pi, m\pi)} = \begin{bmatrix} 0 & (-1)^m \cdot (-1)^{n+1} \\ 0 & 1 \end{bmatrix} \Rightarrow \tau = 0, \Delta = (-1)^{n+m} \Rightarrow \lambda_{1,2} = \pm(-1)^{\frac{n+m+1}{2}} \]

For \(n+m\) odd, \(\lambda_{1,2} = \pm 1\) and we have saddles. For \(n+m\) even, \(\lambda_{1,2} = \pm i\) and we have centers. We can have centers in this system, even though it is nonlinear, because it is reversible.
c) The f.p. are at (x,y)=(-1,-1) and (1,1). The Jacobian is \( J = \begin{bmatrix} y & x \\ 1 & -3y^2 \end{bmatrix} \).

\[ A_1 = J|_{(-1,-1)} = \begin{bmatrix} -1 & -1 \\ 1 & -3 \end{bmatrix} \Rightarrow \tau = -4, \Delta = 4 \Rightarrow \lambda_{1,2} = -2. \]

In a linear system this could be a star or degenerate node. Since the system is nonlinear, it appears to be a stable node.

\[ A_2 = J|_{(1,1)} = \begin{bmatrix} 1 & 1 \\ 1 & -3 \end{bmatrix} \Rightarrow \tau = -2, \Delta = -4 \Rightarrow \lambda_{1,2} = -1 \pm \sqrt{5}. \]

A saddle point.
d) The f.p. is at \((x,y) = (0,0)\). The Jacobian is \(J = \begin{bmatrix} y & 2x \; -1 \end{bmatrix}\). 
\[ A = J_{(0,0)} = \begin{bmatrix} 0 & 0 \; 0 \; -1 \end{bmatrix} \Rightarrow \tau = -1, \; \Delta = 0 \Rightarrow \lambda_{1,2} = 0, -1. \]
In a linear system this would be a non-isolated f.p. The nonlinear plot looks more like a saddle. Although close to the origin it does look similar to a non-isolated f.p., it in fact isn’t. This can be seen by noting that the \(x\) and \(y\) nullclines intersect only at one point.
A Taylor expansion for small $K$, we look near $K=0$ and take

$$F(\theta) = \theta + \Omega - \frac{K}{2\pi} \sin(2\pi \theta)$$

(1)

Assume that for integers $p$ and $q$, $F^q(\theta) = p + \theta$. The winding number is defined as:

$$w = \lim_{n \to \infty} \frac{F^n(\theta_0) - \theta_0}{n}$$

(2)

Let $n = jq + l$, for integers $j$ and $l$:

$$w = \lim_{j \to \infty} \frac{F^{jq+l}(\theta_0) - \theta_0}{jq + l}$$

(3)

Use $F^{jq+l}(\theta_0) = F^l(jp + \theta_0) \approx jp$ in the limit $j \to \infty$ (since $F^l$ can only add a maximum of $l$).

With this it is clear that:

$$w = \lim_{j \to \infty} \frac{F^{jq+l}(\theta_0) - \theta_0}{jq + l} = \frac{p}{q}$$

(4)

The important point here is that we should interpret $q$ as the number of times the function is evaluated before it returns to $\theta_0$ and $p$ as the number of circuits it does as it returns.

For $p=0$, $q=1$, we have: $F(\theta) = \theta \Rightarrow \theta + \Omega - \frac{K}{2\pi} \sin(2\pi \theta) = \theta$. So we have: $\Omega(K) = \frac{K}{2\pi} \sin(2\pi \theta)$. We can only satisfy this relation for any $\theta$ for $|\Omega(K)| \leq \frac{K}{2\pi}$ and since $\Omega \in [0, 1]$ the limit of the tongue is given by $\Omega(K) = \frac{K}{2\pi}$.

For $p=1$, $q=1$, we have: $F(\theta) = 1 + \theta \Rightarrow \theta + \Omega - \frac{K}{2\pi} \sin(2\pi \theta) = 1 + \theta$. So we have: $\Omega(K) = 1 + \frac{K}{2\pi} \sin(2\pi \theta)$. We can only satisfy this relation for any $\theta$ for $\Omega(K) \geq 1 - \frac{K}{2\pi}$ (and again remember $\Omega \in [0, 1]$), so the limit of the tongue is given by $\Omega(K) = 1 - \frac{K}{2\pi}$.

5) For $p=1$, $q=2$, we have: $F^2(\theta) = 1 + \theta$ or:

$$[\theta + \Omega - \frac{K}{2\pi} \sin(2\pi \theta)] + \Omega - \frac{K}{2\pi} \sin(2\pi[\theta + \Omega - \frac{K}{2\pi} \sin(2\pi \theta)]) = 1 + \theta$$

(5)

We look near $K=0$ and take $\Omega = \frac{1}{2} + \varepsilon$ for small $\varepsilon$, so:

$$2\varepsilon - \frac{K}{2\pi} \sin(2\pi \theta) - \frac{K}{2\pi} \sin[2\pi(\theta + \frac{1}{2})] + 2\pi \varepsilon - K \sin(2\pi \theta)] = 0$$

$$2\varepsilon - \frac{K}{2\pi} \sin(2\pi \theta) + \frac{K}{2\pi} \sin[2\pi \theta + 2\pi \varepsilon - K \sin(2\pi \theta)] = 0$$

$$2\varepsilon - \frac{K}{2\pi} \sin(2\pi \theta) + \frac{K}{2\pi} \sin(2\pi \theta) + \frac{K}{2\pi} \cos(2\pi \theta)(2\pi \varepsilon - K \sin(2\pi \theta)) \approx 0$$

(6)

A Taylor expansion for small $K$, $\varepsilon$ has been used. It is clear from the above equation that $\varepsilon = O(K^2)$, so neglecting $\varepsilon$ with respect to $K$:

$$\varepsilon \approx \frac{K^2}{2\pi} \cos(2\pi \theta) \sin(2\pi \theta)$$

$$= \frac{K^2}{8\pi} \sin(4\pi \theta)$$

(7)

(8)

So the limits of the tongue are $\varepsilon = \pm \frac{K^2}{8\pi}$ or $\Omega = \frac{1}{2} \pm \frac{K^2}{8\pi}$. 