Problem Set 2 Solutions

1. The map will be contracting for $|J| < 1$. $|J| = \left| \frac{\partial f}{\partial x} \right| = r(1 - 2x) = r \left| (1 - 2x_0) \right|$. We'd like the map to be contracting for all $x \in [0, 1]$, so we need $|J| < 1$ for $|J| < 1$ at $x = 0$ or $1$. Negative $r$ makes no sense physically, so the map is dissipative for $0 \leq r < 1$.

2. $\dot{X} = r + x - \ln(1 + x)$

3. $X = 0$ near bifurcation $\Rightarrow \dot{X} \approx r + x - (x - \frac{1}{2} x^2) = r + \frac{1}{2} x^2$ looks like a saddle-node!
\[ x = r^2 - x^2 \]

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**Transcritical Bifurcation**

(i)

\[ \frac{dx}{dt} = r - x \]

(ii) Let \( y = x + r \rightarrow \frac{dy}{dt} = -y^2 \]

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\[ \frac{dx}{dt} = x(r - x^2) \]

(iii) \( x \approx x(r - (1 + x)) \approx (r - 1)x - x^2 \)

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\[ r > 1 \]

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\( r = 0 \)

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\( r < 0 \)

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\[ y = 0 \]

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only 1 E.A.
\[ x = r x - \sin x \]  

Supercritical Pitchfork Bifurcation

ii) \[ \dot{x} = r x - (x - \frac{1}{6} x^3) = (r-1)x - \frac{1}{6} x^3 \]

\[ x^* = \frac{r}{2} \pm \sqrt{\frac{r^2}{4} + h} \]

So the saddle node occurs at \( h = \frac{r^2}{4} \)
\[ h > 0 \]

\[ x^* = \frac{c}{2} \pm \sqrt{\left( \frac{c}{2} \right)^2 + h} \]

- no bifurcation will only one sink and one unstable fixed point for all \( r \)

\[ h < 0 \]

\[ x^* = \frac{c}{2} + \sqrt{\left( \frac{c}{2} \right)^2 + h} \]

\[ h = \frac{c}{2} \]

1 stable fixed point
1 unstable fixed point

1 line of saddle-node bifurcation

1 point of transcritical bifurcation

\[ x = h + r x = c + r \quad \text{or} \quad 0 \]

so a perturbation to the saddle-node bifurcation does nothing but shift the bifurcation point. The normal form is unchanged by such a perturbation.
4. Substituting our ansatz into the discretized equation we get:

\[(B^{(n+1)\Delta t})_i \omega \Delta x - B^{(n-1)\Delta t})_i \omega \Delta x)]/\Delta t = -\frac{C}{\Delta x}(B^{(n+1)\Delta t})_i \omega (u(m+1)\Delta x - B^{(n)\Delta t})_i \omega (u(m-1)\Delta x)]\]

(1)

Which can be simplified to:

\[B^\sigma - B^{-\sigma} = -2/\sigma\]

(2)

With \(\sigma = \frac{\Delta t}{\Delta x} \sin(\mu \Delta x)\). This has the solution:

\[B^\sigma = -i\sigma \pm \sqrt{1 - \sigma^2}\]

(3)

Since \( F \propto (B^\sigma)^\gamma \), the scheme will be unstable if \(|B^\sigma| > 1\) for one of the two roots. We have two cases to consider.

First, \(|\sigma| > 1\), so \(B^\sigma\) is pure imaginary. In this case:

\[|B^\sigma| = -\sigma \pm \sqrt{\sigma^2 - 1}\]

(4)

Clearly we will have one unstable root and the scheme will be unstable.

Second, consider \(|\sigma| \leq 1\), so \(B^\sigma\) is complex. In this case:

\[|B^\sigma| = \sigma^2 + 1 - \sigma^2 = 1\]

(5)

And both roots are stable.

We find that if we wish to use the leapfrog scheme without a Robert filter we must choose \(\Delta x\) and \(\Delta t\) carefully so that \(|\sigma| \leq 1\) to avoid exponential divergence of the numerical solution. This behavior has nothing to do with the original equation (anything involving \(\Delta x\) and \(\Delta t\) cannot).