Problem Set 1 Solutions

A.
1. \( \dot{x} = F(x) = 1 - e^{-x^2} \Rightarrow x^* = 0 \quad \text{linearized stability fails.} \)

The graphical approach shows that the fixed point is half-stable.
2. \( \dot{x} = F(x) = ax - x^3 \Rightarrow x^* = 0, \pm \sqrt{a} \quad \frac{dF}{dx} = a - 3x^2, \text{ so:} \)
   \[
   \begin{align*}
   \left. \frac{dF}{dx}\right|_{x=0} &= a \Rightarrow \text{unstable} \\
   \left. \frac{dF}{dx}\right|_{x=\pm \sqrt{a}} &= -2a \Rightarrow \text{stable} \\
   \left. \frac{dF}{dx}\right|_{x=-\sqrt{a}} &= -2a \Rightarrow \text{stable}
   \end{align*}
   
   Notice that the \( x^* = \pm \sqrt{a} \) roots only exist as distinct roots for \( a > 0 \). A supercritical pitchfork bifurcation occurs as \( a \) passes through zero. Notice that linear stability fails at \( a = 0 \).

3. \( \dot{x} = F(x) = x(1 - x)(2 - x) \Rightarrow x^* = 0, 1, 2 \quad \frac{dF}{dx} = 3x^2 - 6x + 2, \text{ so:} \)
   \[
   \begin{align*}
   \left. \frac{dF}{dx}\right|_{x=0} &= 2 \Rightarrow \text{unstable} \\
   \left. \frac{dF}{dx}\right|_{x=1} &= -1 \Rightarrow \text{stable} \\
   \left. \frac{dF}{dx}\right|_{x=2} &= 2 \Rightarrow \text{unstable}
   \end{align*}
   
   \( 1 \)
4. \[ \dot{x} = F(x) = x^2(6 - x) \Rightarrow x^* = 0, 6 \]
\[ \frac{dF}{dx} \bigg|_{x^* = 0} = 0 \Rightarrow ? \]
\[ \frac{dF}{dx} \bigg|_{x^* = 6} = -36 \Rightarrow \text{stable} \]

Using the graphical approach we find that the fixed point at \( x = 0 \) is half-stable.

5. \[ \dot{x} = F(x) = \ln(x) \Rightarrow x^* = 1 \]
\[ \frac{dF}{dx} = \frac{1}{x} \Rightarrow \frac{dF}{dx} \bigg|_{x^* = 1} = 1 \]

The fixed point at \( x = 1 \) is unstable.

B.

1. \[ x^* = x^*(1 - x^*) \Rightarrow x^*(x^* + (1 - r)) = 0 \Rightarrow x^* = 0, 1 - \frac{1}{r} \]
\[ \frac{dF}{dx} = r(1 - 2x) \Rightarrow \frac{dF}{dx} \bigg|_{x^* = 0} = r \text{ and } \frac{dF}{dx} \bigg|_{x^* = 1 - \frac{1}{r}} = 2 - r \]

A fixed point is stable if \( \left| \frac{dF}{dx} \right| < 1 \) and unstable if \( \left| \frac{dF}{dx} \right| > 1 \). We’ll only consider \( r > 0 \) on physical grounds.

The fixed point \( x^* = 0 \) is stable for \( 0 \leq r < 1 \) and unstable for \( r > 1 \). At \( r = 1 \) we’ll need to use graphical methods.

The fixed point \( x^* = 1 - \frac{1}{r} \) is stable for \( 1 < r < 3 \) and unstable for \( r > 3 \). For \( 0 \leq r < 1 \) this fixed point is unphysical. At \( r = 1 \) both fixed points overlap. At \( r = 3 \) we’ll need to use graphical methods.

It’s hard to see what’s happening at \( r = 3 \) with a cobweb diagram, so we’ll use a timeseries. Fig. 1 shows that at \( r = 3 \) the fixed point at \( x^* = 1 - \frac{1}{r} \) is very slightly stable. We would have to expand to \( O(\eta^2) \) to see this analytically. The fixed point at \( x^* = 0 \) is clearly stable for \( r = 1 \).

2. Fig. 2 shows the timeseries plots for the different values of \( r \). At \( r = 0.4 \) the system approaches the fixed point at \( x = 0 \). At \( r = 2 \) the system monotonically approaches the fixed point at \( x^* = 1 - \frac{1}{r} \).
At $r=2.8$ and $r=2.9$ this fixed point is approached in an oscillatory manner. At $r=3$ a flip bifurcation occurs and for $r=3.2$ we have a 2-cycle. The system undergoes the period-doubling route to chaos and at $r=4$ there is chaotic behavior.

C. For large $n$ we expect the system to be very near the fixed point at $x=0$. This means $x_n \ll 1$ so we have: $x_{n+1} = r x_n (1 - x_n) \approx r x_n$. The solution to this is $x_n = (\text{const.}) r^n$. The constant is determined by when the system enters this regime.
Figure 2: