Nonlinear dynamics and chaos

Applied Mathematics 147

(Fall 2004)

Instructor: Eli Tziperman
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Day and time of course: Tue Thu 11:30-1;
Location: Cruft 319
Regular section time: Monday 17:00-18:00. Cruft 319
TF office hours for Dorian: Wednesday 14:00-15:00, Museum Building room 101, or call/ email him (Dorian Abbot ¡abbot@fas.harvard.edu¿)

Contents

Announcements Last updated: Jan 15, 2005.
Final exam: will be held on Tue 01/18 at 2:15
Room: Sever Hall 102
Exam Aids Allowed: ONLY the following: 1. your notes from class; 2. notes from course website; 3. section handouts; 4. problem sets and solutions (your own and those posted on the course home page); 5. calculators/ graphing calculators.
Feel free to write or call me with any questions

Teaching notes online:
1_intro, 2_intro, 3_bif1d, 4_bif1d, 5_bif1d, 6_bif2d, 7_bif2d, 8_bif2d, 9_bif2d, 10_chaos, 11_chaos, 12_chaos, 13_chaos, 14_chaos, 15_fract, 16_hamilt, 17_hamilt,
Sample Matlab programs: bakers_map.m, circle_map.m, cobweb_func.m, driven_pendulum.m, euler_course.m, feigenbaum.m, glider.m, henon.m, logistic_map.m, lorenz1.m, lorenz2.m, mandelbrot_Setsv.m, modified_euler_1d.m, my_quiver.m, pendulum.m, pendulum_self_sustained.m, perturbation_series_example.m pplane6.m, ppn6out.m, recon_lorenz_phase_space.m, sea_ice_switch.m, spectrum_harmonics.m, shilnikov.m, standard_map.m, standard_map_interactive.m, van_der_pol.m,
Homework: (due in class one week from date given) 01, 02, 03, 04, 05, 06, 07, 08, 09, 10, 11, 12
What's the point about optional/ extra credit problems: apart from the fun of doing them, they will count against homework problems in which you may have missed an answer. If you don't do the challenge problems, make sure you understand their solutions once posted.
homework solutions: 01, 02, 03, 04, 05, 06, 07, 08, 09, 10, 11 12
Section notes from Dorian: 01, section1, section2, section3, section4, section5, section6, section7, section8, Gaspard_letter.pdf, section9, section10, section11,

1 Textbooks:

- Mostly this one: (**St**) *Nonlinear Dynamics and Chaos: With Applications to Physics, Biology, Chemistry, and Engineering* by Steven H. Strogatz
- and a bit of this: (**Sc**) *Deterministic Chaos: An Introduction*; Heinz Georg Schuster, [VCH, 2nd edition, 1989]
- and this: (**Ott**) *Chaos in dynamical systems*, 1993. Edward Ott, Cambridge University Press.

Additional reading:

- (**GH**) *Nonlinear Oscillations, Dynamical Systems and Bifurcations of Vector Fields*, Guckenheimer, J and P. Holmes, Springer-Verlag, 1983.
- (**W**) *Introduction to Applied Nonlinear Dynamical Systems and Chaos.* Stephen Wiggins, 1990. (Texts in Applied Mathematics, Vol 2).

- (**JS**) *Classical Dynamics, a contemporary approach.* Jorge V. Jose and Eugene J. Saletan. *1993 Cambridge University Press.*

- (**G**) *Classical Mechanics*, Herbert Goldstein, 2nd edition, 1981. Addison Wesley.

2 Outline

The course will introduce the students to the basic concepts of nonlinear physics, dynamical system theory, and chaos. These concepts will be demonstrated using simple fundamental model systems based on ordinary differential equations and some discrete maps. Additional examples will be given from physics, engineering, biology and major earth systems. The aim of this course is to provide the students with analytical methods, concrete approaches and examples, and geometrical intuition so as to provide them with working ability with non-linear systems.
The following detailed outline is very preliminary, will likely change and include less material.

2.1 Introduction

(1 week) (**St** 1-37,+)

- A bit of history (Lorentz and the "butterfly effect")

- Modeling - defining phase space, dimension, parameters, deterministic versus stochastic modeling finite vs infinite dimensional (PDE's, integral eq.) models, linear vs non-linear, autonomous vs non-autonomous systems

- Examples: population dynamics, pendulum, Lorenz eq., ...

- The geometric approach to dynamical systems

- Fixed points, linearization, and stability

- Non-dimensionalization, the Buckingham Pi theorem (see notes here), small parameters, scales.

- Dynamical systems - continuous vs discrete time (ODEs vs maps; **St** 348), conservative vs dissipative (**St** 312).

- Existence, uniqueness and smooth dependence of solutions of ODE's on initial conditions and parameters.
- The role of computers in nonlinear dynamics, a simple example of a numerical solution method for ODEs (improved Euler scheme).
- Outline of rest of course.

2.2 Bifurcations in one dimensional systems

(3 weeks)

- What's a bifurcation, local vs global bifurcations (**GH** §3.1). Implicit function theorem, classification of bifurcations by number and type (real/ complex) eigenvalues that cross the imaginary axis.
- saddle-node bifurcation (**St** §3.1; **GH** §3.4)
- Transcritical bifurcation, super critical and sub critical (**St** §3.2; **GH** §3.4).
- Pitchfork, super-critical and sub-critical. bead on a rotating hoop, higher order nonlinear terms and hysteresis (**St** §3.4; **GH** §3.4)
- Some generalities: center manifold and normal form. (**GH** §3.2-3.3).
- Role of symmetry and symmetry breaking (imperfect bifurcations), relation to catastrophes and sudden transitions. (**St** §3.6)
- Flows on a circle - oscillators, synchronization (fireflies flashing, Josephson junctions) (**St** §4)

2.3 Two-dimensional systems and some more basics

(4 weeks)

- Linear systems: classifications, fixed points, stable and unstable spaces (**St** §5)
- Non-linear systems: phase portrait (**St** §6.1), fixed points and linearization(**St** §6.3), stable and unstable manifolds (**St** §6.4), conservative systems (**St** §6.5), reversible systems (**St** §6.6), Solution of the (fully non-linear) damped pendulum equation (**St** §6.7), index theory (**St** §6.8).

- Limit cycles: Ruling out and finding out closed orbits (Lyapunov functions, Poincare Bendixon theorem) (**St** §7.2 and §7.3)
- relaxation oscillations (relation to glacial cycles) (**St** §7.5), weakly non-linear oscillators (Duffing eq) (**St** §7.6), Averaging method and two time-scales (**St** §7.6)
- Hopf bifurcation and oscillating chemical reactions (**St** §8.2),
- Global bifurcations of cycles: saddle-node infinite period, and homoclinic bifurcations, examples in Josephson Junction and driven pendulum in 2D (**St** §8.4 and§8.5)
- Quasi periodicity, coupled oscillators, nonlinear resonance/ frequency locking (Frequency locking of glacial cycles to earth orbital variations), (**St** §8.6)

2.4 Chaos, transition to chaos

(4 weeks)
The Lorentz model as an introduction to chaotic systems (examples briefly motivating it from atmospheric dynamics and as a model of Magnetic field reversals of the Earth); and then a more systematic characterization of chaotic systems (examples from fluid dynamics and mantle convection) (**St** §9). Some preliminaries: Poincare maps.
Universal routes to chaos:

- Period doubling: logistic map, chaos, periodic windows, renormalization, quantitative and qualitative universality. (**Sc** §3)
- Intermittency: in Lorenz system, in logistic map. Length of laminar intervals from renormalization and simpler approaches. Categories of intermittency (types I,II,III), (**Sc** §4).
- Quasi-periodicity/ 1-2-chaos/ Ruelle-Takens-Newhouse; breakdown of 2d torus; in experimental systems; 1D circle map and overlapping of resonances; reconstructed circle map from a time series; damped-forced pendulum and El Nino's chaos (**Sc** §6)

More:

- Characterizing chaotic systems: Delay coordinates, embedding, Lyapunov exponents (Ott §4.4 p. 129); Kolmogorov entropy (**Sc** Appendix F and p 113; Greiner, Neise and Stocker "*thermodynamics and statistical mechanics*", p. 150); fractals and fractal dimensions, dimension spectrum (**St** §11, p. 398-412; **Ott** §3, p69-71, 78-79, 89-92); Multi-fractals: dissipation in a turbulent flow, relation to dimension spectrum. (**Ott** §9, p 305-309).

- The horseshoe map and symbolic dynamics (**Ott** 108-114); Heteroclinic and homoclinic tangles and creation of a horseshoe from a homoclinic intersection (**Ott** §4.3). Shilnilov's phenomenon and chaos due to a 3d homoclinic orbit (GH, §6.5, p 318-323; and p 12-14 in Vered Rom-Kedar's notes).

2.5 Chaos in Hamiltonian systems

(time permitting)

- Examples (Pendulum, The n-body problem)

- Basics: Hamiltonian systems; Liouville theorem/ symplectic condition; (**Ott** §7.1.1-7.1.2 p 208-215).

- Motivation: the kicked rotor and chaos in the standard map (**Ott**, p 216-217, 235-237; **JS** §7.5.1 p. 453-459).

- More Basics: integrable vs non-integrable Hamiltonian systems; motion of integrable on N-torus; Canonical change of coordinates and generating functions; (**G**, §9-1, p. 378-385, **Ott** §7.1.1-7.1.2 p 208-215).

- Perturbations to integrable systems; averaging; resonant and non-resonant tori (**G**, §11-5, p 519-523); destruction of resonance tori and arising of chaos, KAM theory (**Ott** §7.2).

- "diffusion" (**Ott** §7.3.3), fluid mixing (**Ott** p 246-249).

2.6 Misc

(time permitting)

- Controlling chaos

3 Course requirements

Homeworks will be given throughout the course. The best 80% of the assignments will constitute 50% of the final grade. A final exam (possibly a take home) will constitute another 50%.

4 Misc

1. An interesting account of Poincare's entry for King Oscar II 3-body problem competition, his error, and his discovery of sensitivity to initial conditions: here.
2. A nice on-line chaos course is at http://www.cmp.caltech.edu/ mcc/Chaos_Course/
3. Also nice: an interactive on line demo of a driven pendulum.
4. Devil's staircase in circle map and Farey tree:
5. For some interesting details about the KAM theorem, check here.
6. On the Mandelbrot set