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APM147
12/9/04

Problem Set 12 Solutions

1. The time series is plotted for $\dot{\theta}$ instead of θ itself because θ may become very large if the pendulum winds around the top and it may be hard to understand the system's behavior.

   (a) For k=0.5 we see similar behavior to the transition to chaos in the circle map. We can enter and leave mode-locked regions (Arnold tongues) and have quasi-periodic behavior between them. After the onset of chaos we can still get mode-locked regions.

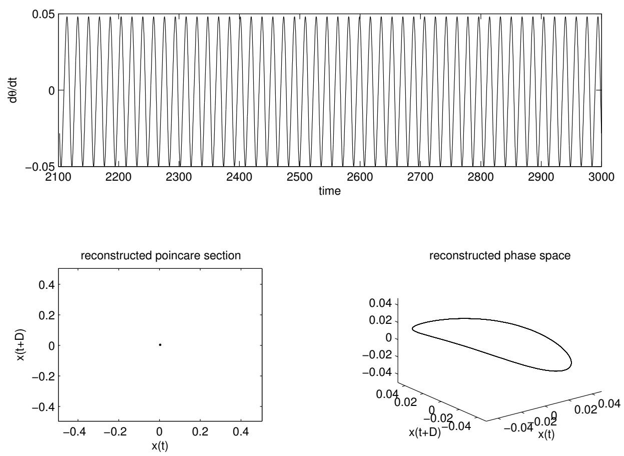

Figure 1: f=0.1, a mode-locked solution (periodic).

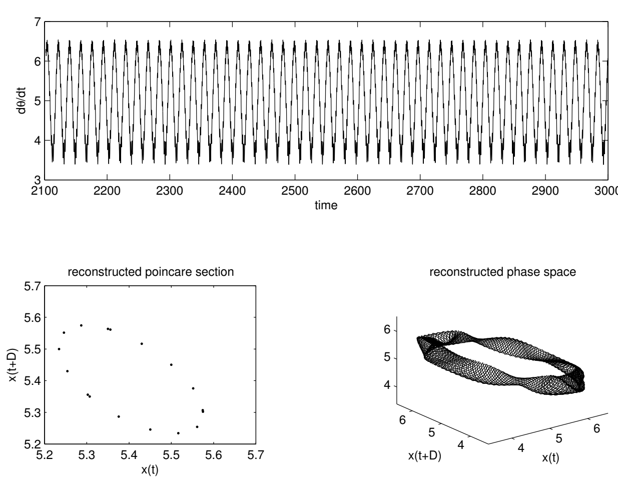

Figure 2: f=0.5, a quasi-periodic solution. The bagel in phase space and circle in the Poincare section signify this.

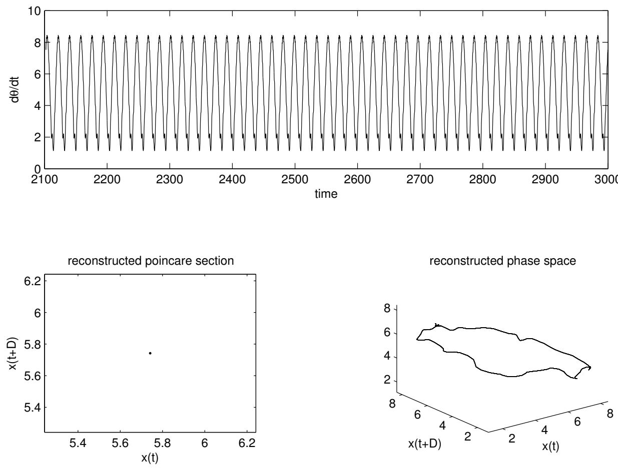

Figure 3: f=1.25, we enter another mode-locked solution. Interestingly, from here we follow a period-doubling route to chaos.

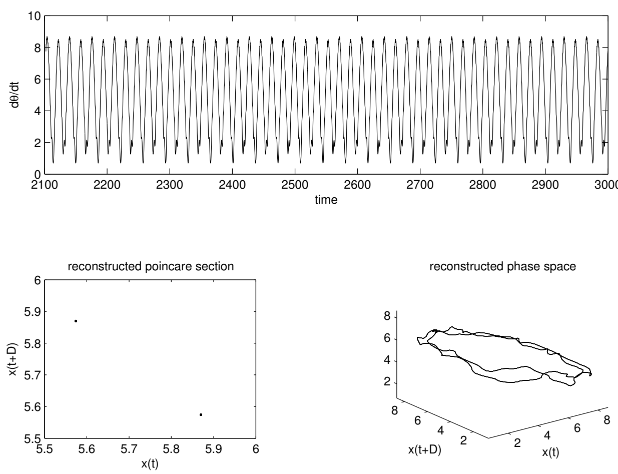

Figure 4: f=1.3, the period has doubled.

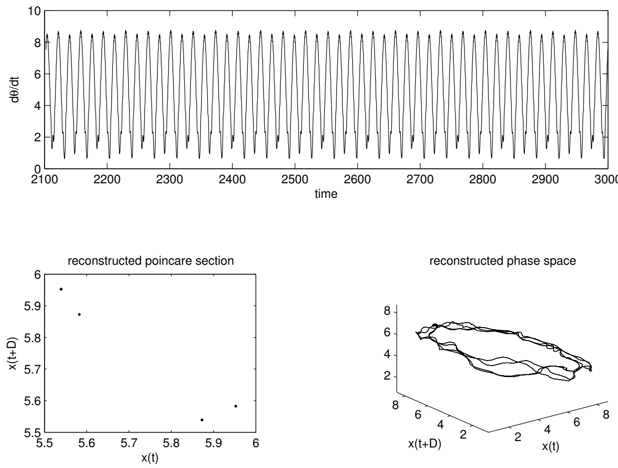

Figure 5: f=1.31, the period has doubled again.

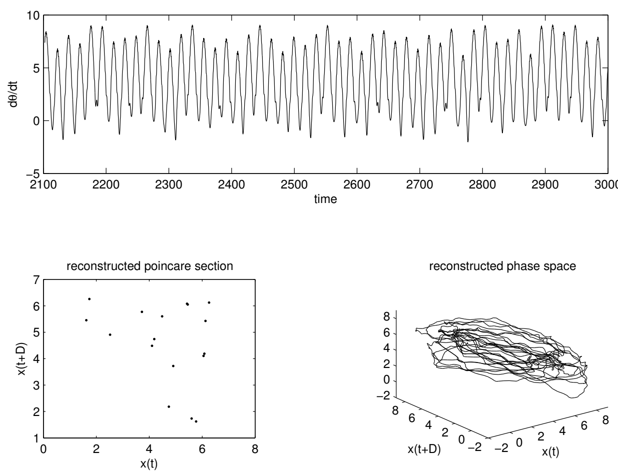

Figure 6: f=1.4, chaos! If we keep increasing f we will reach a phase-locked solution near f=1.9 that will go through another period-doubling cascade back to chaos.

(b) When k=0, we don't find any quasi-periodic behavior. So we might consider this a degenerate case of the quasi-periodic route to chaos. The quasi-periodicity is lost due to the symmetry gained by setting k=0.

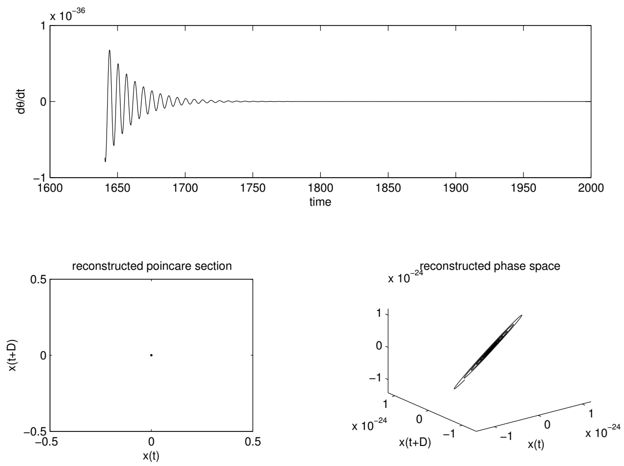

Figure 7: For small forcing (f=0) the system is damped.

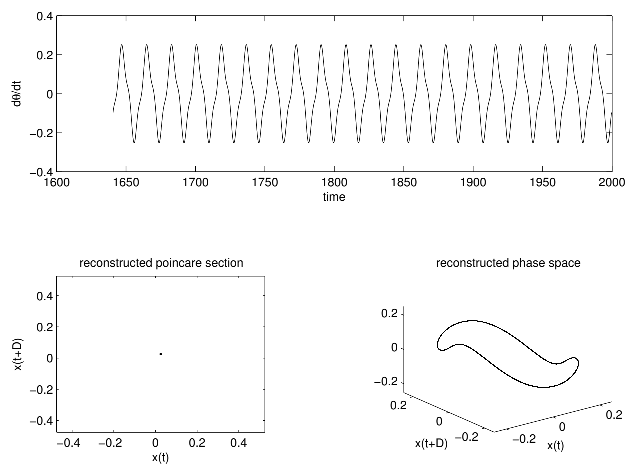

Figure 8: At larger forcing (f=0.5) we get mode-locked behavior.

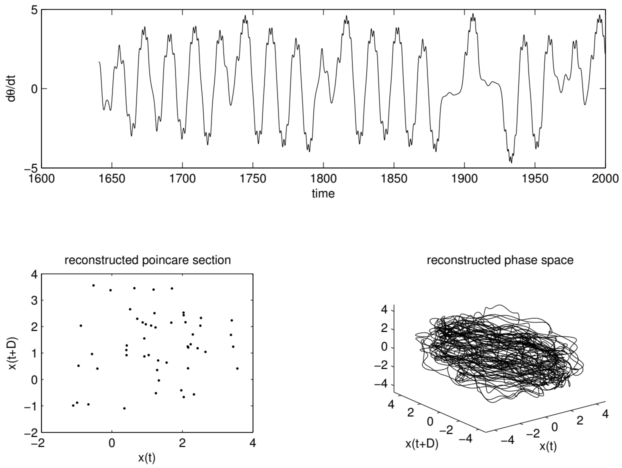

Figure 9: Then the system becomes chaotic (f=1).

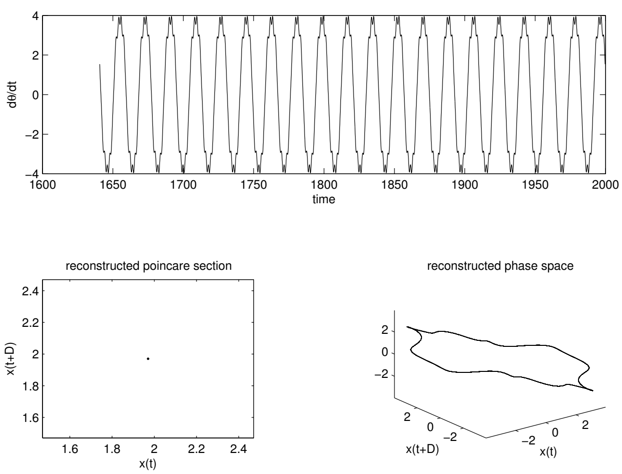

Figure 10: You can find mode-locked periodic behavior after the onset of chaos (f=1.1 here).

(c) I was able to get a very rough picture of what was going on using this method. A major annoyance is that the behavior seems to depend pretty strongly on the time-stepping. When I decreased the time-stepping by a factor of 10 the behavior changed dramatically. To give you an idea of how complex an actual state plot of a system like this is, here's one from (D'Humieres *et al.*, 1982) for k=0.

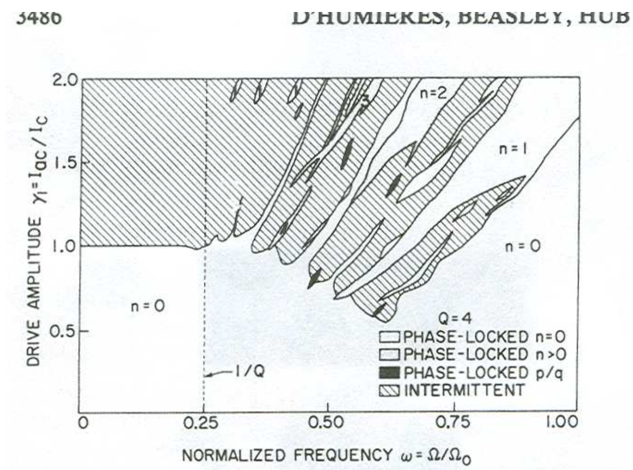

FIG. 3. State diagram for the driven pendulum with $Q=4$ and $\gamma_0=0$.

Figure 11: The x-axis is the frequency of the driving and the y-axis is the coefficient of the periodic forcing.

2. In general when the map of an experimental system that is similar to the circle map becomes non-invertible, chaos occurs.

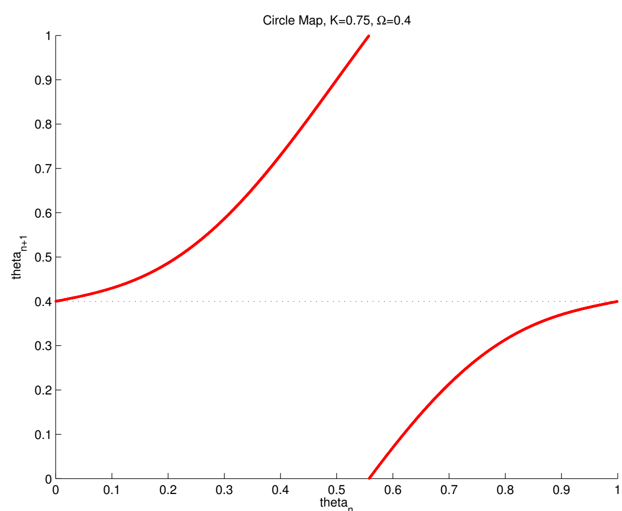

Figure 12: For K<1 θ_{n+1} is a single-valued function of θ_n and vice versa, so the circle map is invertible for K<1.

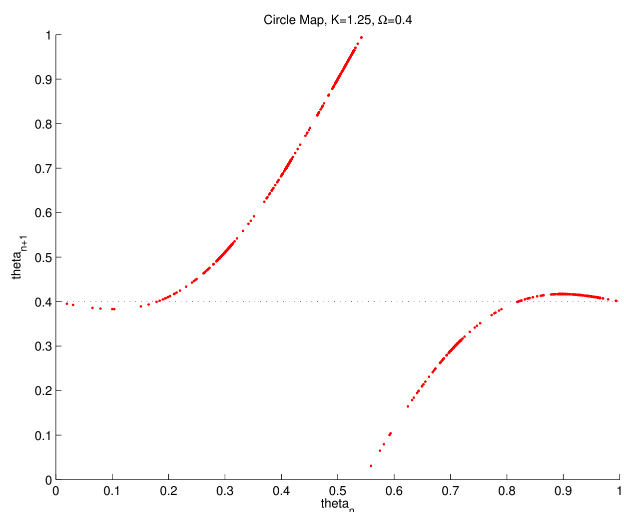

Figure 13: For K>1 θ_{n+1} is a single-valued function of θ_n, but θ_n is not a single-valued function of θ_{n+1}. so the circle map is not invertible for K>1. I had to use 30 random starting points the get a decent picture of the function here.

3. The nth iteration of the Koch curve has 4^n intervals of length $(\frac{1}{3})^n$. So the box dimension is:

$$\begin{aligned} D_{box} &= \lim_{\varepsilon\to\infty} \frac{ln(N(\varepsilon))}{ln(\frac{1}{\varepsilon})} \\ &= \lim_{n\to\infty} \frac{ln(4^n)}{ln(3^n)} \\ &= \frac{ln(4)}{ln(3)} \\ &\approx 1.26 \end{aligned} \tag{1}$$

4. The nth iteration has 3^{2n} 2D boxes each with side length $\varepsilon = (\frac{1}{3})^n$. 4^n are filled. The box dimension is:

$$\begin{aligned} D_{box} &= \lim_{\varepsilon\to\infty} \frac{ln(N(\varepsilon))}{ln(\frac{1}{\varepsilon})} \\ &= \lim_{n\to\infty} \frac{ln(4^n)}{ln(3^n)} \\ &= \frac{ln(4)}{ln(3)} \\ &\approx 1.26 \end{aligned} \tag{2}$$

The same as the Koch curve!

5. We have a 3D cube, such that divide into 27, we obtain $27 = 3^3$ cubes. After one iteration, we removed $2 \cdot 3 + 1$ cubes ($2 \cdot 3$ faces and 1 center). We are left with $[3^3 - (2 \cdot 3 + 1)]$ cubes. After 2 iterations $[3^3 - (2 \cdot 3 + 1)]^2$... After n iterations, $[3^3 - (2 \cdot 3 + 1)]^n$ each with side length $\varepsilon = (\frac{1}{3})^n$. The box dimension is:

$$\begin{aligned} D_{box} &= \lim_{\varepsilon\to\infty} \frac{ln(N(\varepsilon))}{ln(\frac{1}{\varepsilon})} \\ &= \lim_{n\to\infty} \frac{ln[3^3 - (2 \cdot 3 + 1)]^n}{ln(3^n)} \\ &= \frac{ln(20)}{ln(3)} \\ &\approx 2.73 \end{aligned} \tag{3}$$

6. Now imagine we have an N-dimensional cube and we keep the only the corners at each fractal iteration. We still have $\varepsilon = (\frac{1}{3})^n$ for the nth iteration. So the box dimension is:

$$\begin{aligned} D_{box} &= \lim_{\varepsilon\to\infty} \frac{ln(N(\varepsilon))}{ln(\frac{1}{\varepsilon})} \\ &= \lim_{n\to\infty} \frac{ln[3^N - (2 \cdot N + 1)]^n}{ln(3^n)} \\ &= \lim_{n\to\infty} \frac{ln[3^N - (2 \cdot N + 1)]}{ln(3)} \end{aligned} \tag{4}$$

7. There are 4^{2n} 2D boxes at the nth iteration, so $\varepsilon = (\frac{1}{4})^n$. 8^n of these boxes are filled. So the box dimension is:

$$\begin{aligned} D_{box} &= \lim_{\varepsilon \to \infty} \frac{ln(N(\varepsilon))}{ln(\frac{1}{\varepsilon})} \\ &= \lim_{n \to \infty} \frac{ln(8^n)}{ln(4^n)} \\ &= \frac{ln(8)}{ln(4)} \\ &= \frac{3}{2} \end{aligned} \tag{5}$$