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APM147
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Problem Set 9 Solutions

1. I played with a simple MATLAB script to try to determine where the period doubling bifurcations occurred manually. This is not a particularly accurate way of doing this because near bifurcation points convergence is slow and round-off errors are a big problem. Because of these issues, this is not the way people actually try to determine δ. The eigenvalue method from Feigenbaum, involving taking advantage of super-stable cycles, is usually still used. See feigenbaum.m on the website for an example.

   For the sine map, I found $r_1 \approx 0.713,\ r_2 \approx 0.831,\ r_3 \approx 0.858$ so $\delta \approx \frac{0.831-0.713}{0.858-0.831} \approx 4.4$. We know this is a quadratic map, and this estimate is relatively close to the value $\delta = 4.669\ldots$.

   For the quartic map given, I found $r_1 \approx 0.746,\ r_2 \approx 1.113,\ r_3 \approx 1.161$ so $\delta \approx \frac{1.113-0.746}{1.161-1.113} \approx 7.6$. (Briggs, 1990) give $\delta = 7.28\ldots$ for quartic maps, so again the estimate is not too bad.

2. *This problem uses Schuster's notation, in which α is a positive number. In Strogatz α is a negative number. This is why equation 3.22 in Schuster looks slightly different from equation (2) in section 10.7 of Strogatz.*

   (a) Assume g(x) is a fixed point of the doubling transformation, so:

   $$g(x) = -\alpha g[g(\frac{x}{-\alpha})] \equiv T[g] \tag{1}$$

   Multiply by μ and let x=$\frac{x}{\mu}$:

   $$\mu g(\frac{x}{\mu}) = -\alpha\mu g[g(\frac{x}{-\alpha\mu})] \tag{2}$$

   Rearrange inside of function:

   $$\mu g(\frac{x}{\mu}) = -\alpha\mu g[\frac{1}{\mu}(\mu g(\frac{x}{-\alpha\mu}))] = T[\mu g(\frac{x}{\mu})] \tag{3}$$

   So $\mu g(\frac{x}{\mu})$ is also a fixed point of the doubling transformation.

   (b) By definition, g(x)=$-\alpha g^2(\frac{x}{-\alpha})$. So we must have $g(-\alpha x) = -\alpha g^2(x)$ for any x. If x^* is a fixed point of g(x), then $g^2(x^*) = x^*$ so $g(-\alpha x^*) = -\alpha x^*$ and $-\alpha x^*$ is also a fixed point of g(x).

   This means that if g(x) has a single fixed point, then it must have an infinite number of them.

   g(0)=1 and g(1)=$-\frac{1}{\alpha}$<1 so assuming g is well-behaved it must have one fixed point between x=0 and 1. This means it must have an infinite number of fixed points and so crossings of the line y=x.

   We expect the g to be an even function of x. So we also expect an infinite number of crossings of the line y=-x.

(c) Approximate g(x)=1+c_2x^2. So we have:

$$\begin{aligned} g(x) &= -\alpha g^2(\frac{x}{-\alpha}) \\ 1+c_2x^2 &= -\alpha[1+c_2(1+c_2(\frac{x}{-\alpha})^2)^2]+O(x^4) \\ 1+c_2x^2 &= -\alpha(1+c_2)-2\frac{c_2^2}{\alpha}x^2+O(x^4) \end{aligned} \tag{4}$$

To satisfy this equation at for all x up to O(x^2), we need:

$$1+c_2=\frac{1}{-\alpha}\; and\; c_2=\frac{-\alpha}{2} \tag{5}$$

This leads to a quadratic in α (or c_2). If we take the positive root we get $\alpha = 1+\sqrt{3} \approx 2.73$ so $c_2 \approx -1.37$. Not to shabby for so little work!

3. Following from Schuster, page 46:

$$\begin{aligned} g_{i-1}(x) &\equiv \lim_{n\to\infty}(-\alpha)^n f_{R_{n+i-1}}^{2^n}[\frac{x}{(-\alpha)^n}]\;(by\;definition) \\ &= \lim_{n\to\infty}(-\alpha)(-\alpha)^{n-1} f_{R_{n+i-1}}^{2^{n-1+1}}[\frac{1}{(-\alpha)}\frac{x}{(-\alpha)^{n-1}}]\;(rearrangement) \\ &= \lim_{m\to\infty}(-\alpha)(-\alpha)^m f_{R_{m-1}}^{2^{m+1}}[\frac{1}{(-\alpha)}\frac{x}{(-\alpha)^m}]\;(let m=n-1) \\ &= \lim_{m\to\infty}(-\alpha)(-\alpha)^m f_{R_{m+i}}^{2\cdot 2^m}[\frac{1}{(-\alpha)}\frac{x}{(-\alpha)^m}]\;(rearrange\;exponent) \\ &= \lim_{m\to\infty}(-\alpha)(-\alpha)^m f_{R_{m+i}}^{2^m}[f_{R_{m+i}}^{2^m}[\frac{1}{(-\alpha)}\frac{x}{(-\alpha)^m}]]\;(what\;we\;mean\;by\;f^2) \\ &= \lim_{m\to\infty}(-\alpha)(-\alpha)^m f_{R_{m+i}}^{2^m}[\frac{1}{(-\alpha)^m}(-\alpha)^m f_{R_{m+i}}^{2^m}[\frac{1}{(-\alpha)}\frac{x}{(-\alpha)^m}]]\;(clearly) \\ &= -\alpha g_i[g_i(-\frac{x}{\alpha})]\;(by\;definition) \end{aligned}$$

4. This is directly out of Strogatz. If you have a question about it please see me. Mathematica helps with the algebra.

5. In the quartic case we expect $g_1(x) = 1+\sum_{i=1}^{n} c_i x^{4i}$. If we keep only the first term, we find:

$$\begin{aligned} g(x) &= -\alpha g^2(\frac{x}{-\alpha}) \\ 1+c_1x^4 &= -\alpha[1+c_1(1+c_1(\frac{x}{-\alpha})^4)^4]+O(x^8) \\ 1+c_1x^4 &= -\alpha(1+c_1)-4\frac{c_1^2}{\alpha^3}x^4+O(x^8) \end{aligned} \tag{6}$$

Which gives two equations:

$$\begin{aligned} -\frac{1}{\alpha} &= 1+c_1 \\ 1 &= -4\frac{c_1}{\alpha^3} \end{aligned} \tag{7}$$

These yield a quartic equation for α: $\alpha^4 - 4\alpha - 4 = 0$. Using 'fzero' in MATLAB I find the solution to be $\alpha \approx 1.835$. (Briggs, 1990) gives $\alpha = 1.690\ldots$ for the quartic case, so this method gets us within 10%.

6. From section notes 8. I'll explain the plots as I did in section.

   For certain values of r the Lorenz system exhibits "windows of periodic behavior." For the standard choices of b=$\frac{8}{3}$ and σ=10, a "period doubling cascade" to chaos occurs as r decreases for r just below 100.

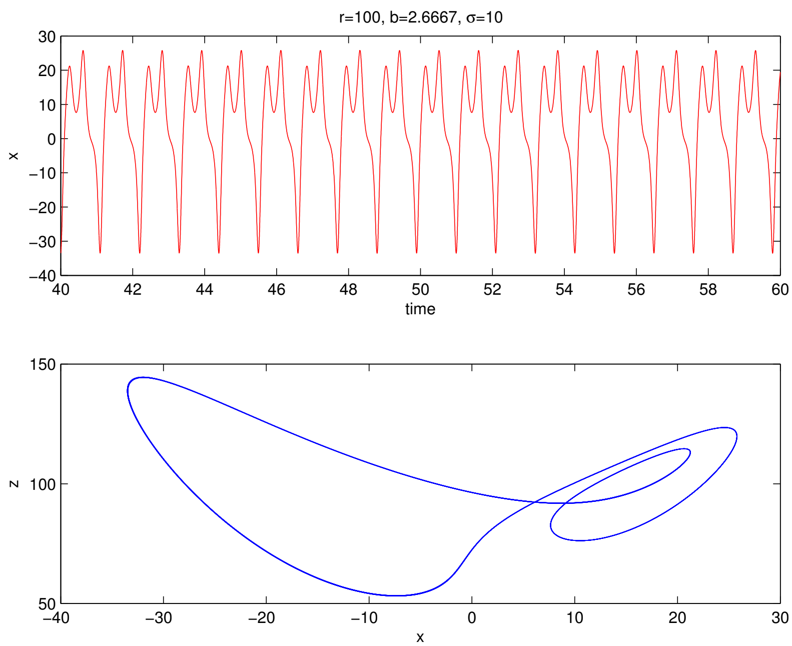

Figure 1: For r=100 we have a limit cycle. Notice that if we were randomly trying r values and happened to try r=100 (which might be a common choice) we would see periodic behavior in the middle of chaos.

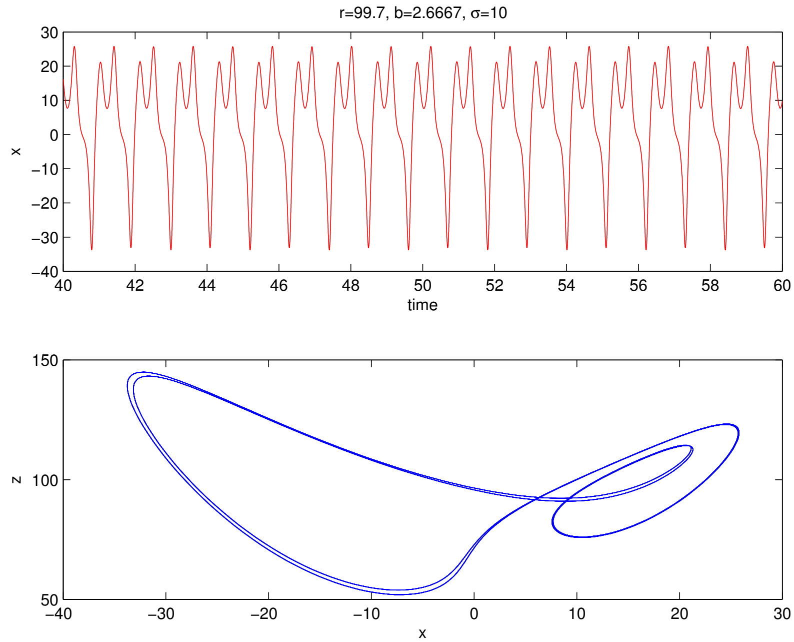

Figure 2: A period doubling bifurcation has occurred. Notice that this cycle is very close to two limit cycles for r=100. In fact, I wouldn't be able to tell the difference except from the time series plots, only from the plots in phase space.

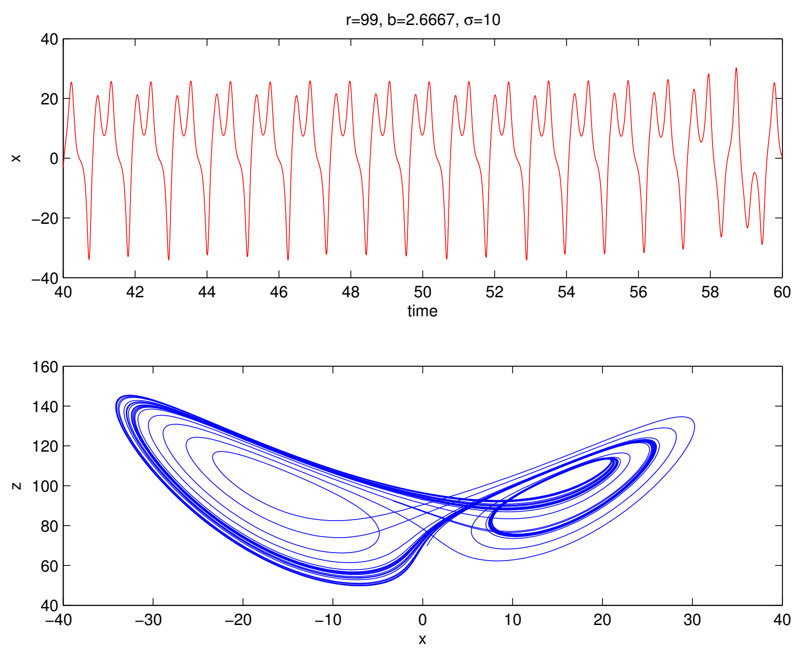

Figure 3: Here the behavior is chaotic, but for some time it almost behaves like the limit cycle for r=100. So if we start at r=90 or so and increase r toward r=100 we would see what looks like a limit cycle start to materialize from chaos.

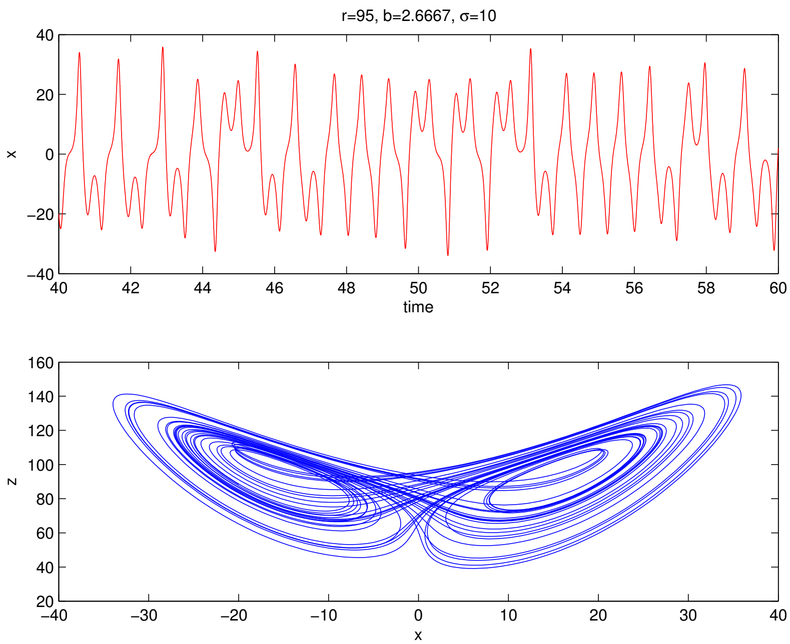

Figure 4: Even at r=95 there is structure that looks like the r=100 limit cycle near t=50.

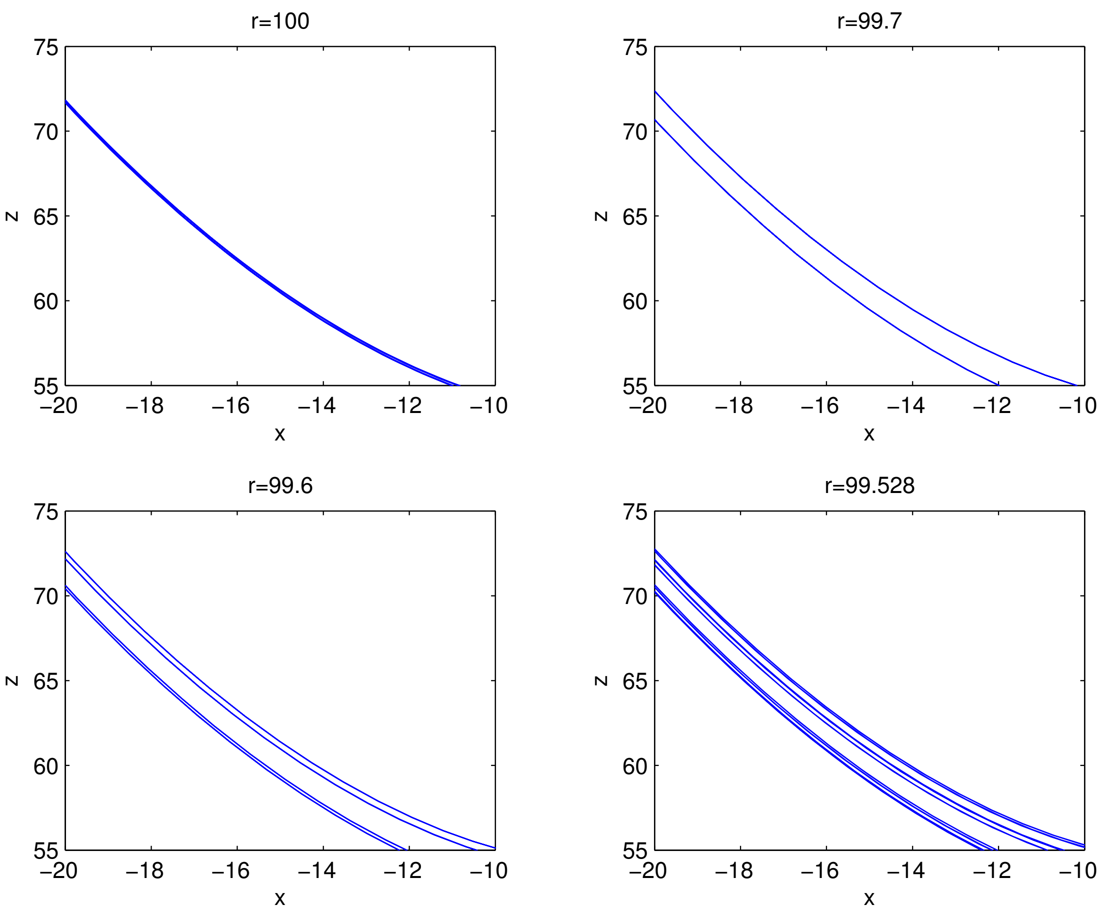

Figure 5: Here I've zoomed in on one region of phase space to show 3 period doublings. It is very hard to see the period doublings from a time series or a larger region in phase space. Notice that the doublings start happening very close together as r decreases.