Homework #7
Nonlinear dynamics and chaos

1. Regular perturbation problem: use second order perturbation theory (up to and including $O(\varepsilon^2)$) to find approximations to one of the roots of each of the following two equations:

$$x^3 - \varepsilon x - 1 = 0 \quad (1)$$
$$x^3 + \varepsilon x^2 - x = 0 \quad (2)$$

set $\varepsilon = 0.001$ and compare your approximate results to the numerical results at $O(1), O(\varepsilon), O(\varepsilon^2)$ calculated using the Matlab "roots" command.

2. Hopf bifurcation: (Strogatz 8.2.1) consider the biased van der Pol oscillator $\ddot{x} + \mu(x^2 - 1)\dot{x} + x = a$. Find the curves in the (μ, a) plane at which Hopf bifurcation occur.

3. (Strogatz 8.2.2-4) By calculating the linearization at the origin, show that the system

$$\dot{x} = -y + \mu x + xy^2 \quad (3)$$
$$\dot{y} = x + \mu y - x^2 \quad (4)$$

has pure imaginary eigenvalues when $\mu = 0$.

4. Plot the phase portraits for the system in the previous section using the Matlab quiver command for different values of μ and show that the system under goes a Hopf bifurcation at $\mu = 0$. is it subcritical, supercritical or degenerate?

5. (a) Rewrite the system from the above two equations in polar coordinates;
   (b) Show that if $r \ll 1$, then $\dot{\theta} \approx 1$ and $\dot{r} \approx \mu r + \frac{1}{8} r^3 + \ldots$ where the terms omitted are oscillatory and have essentially zero time-average around one cycle.
   (c) Show that the formula from (b) suggest the presence of an *unstable* limit cycle of radius $\approx \sqrt{-8\mu}$ for $\mu < 0$ (valid only for $r \ll 1$ and therefore for $|\mu| \ll 1$). Which Hopf bifurcation is this therefore?