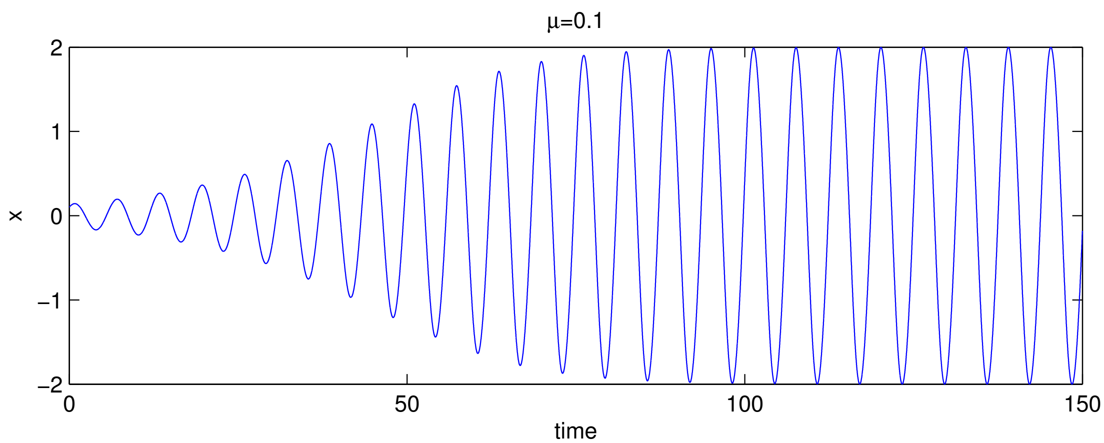
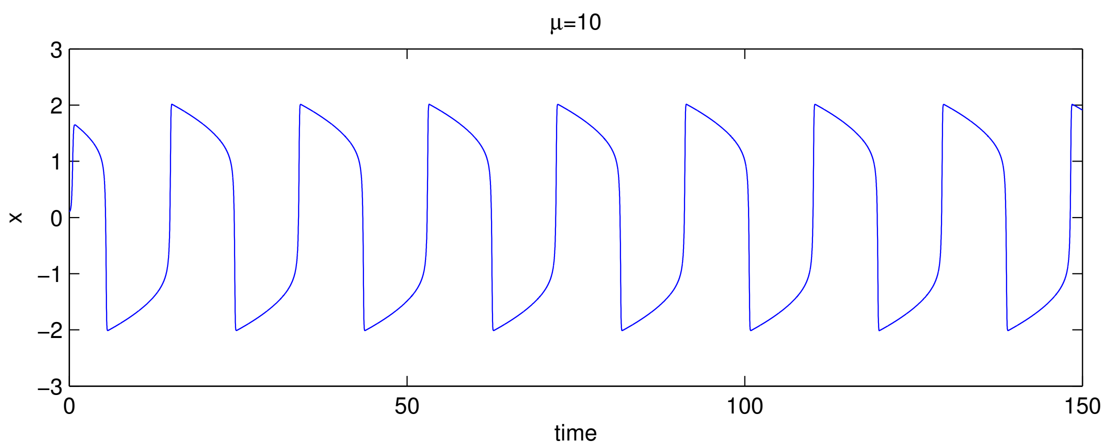
Dorian Abbot
APM147
11/4/04

Problem Set 6 Solutions

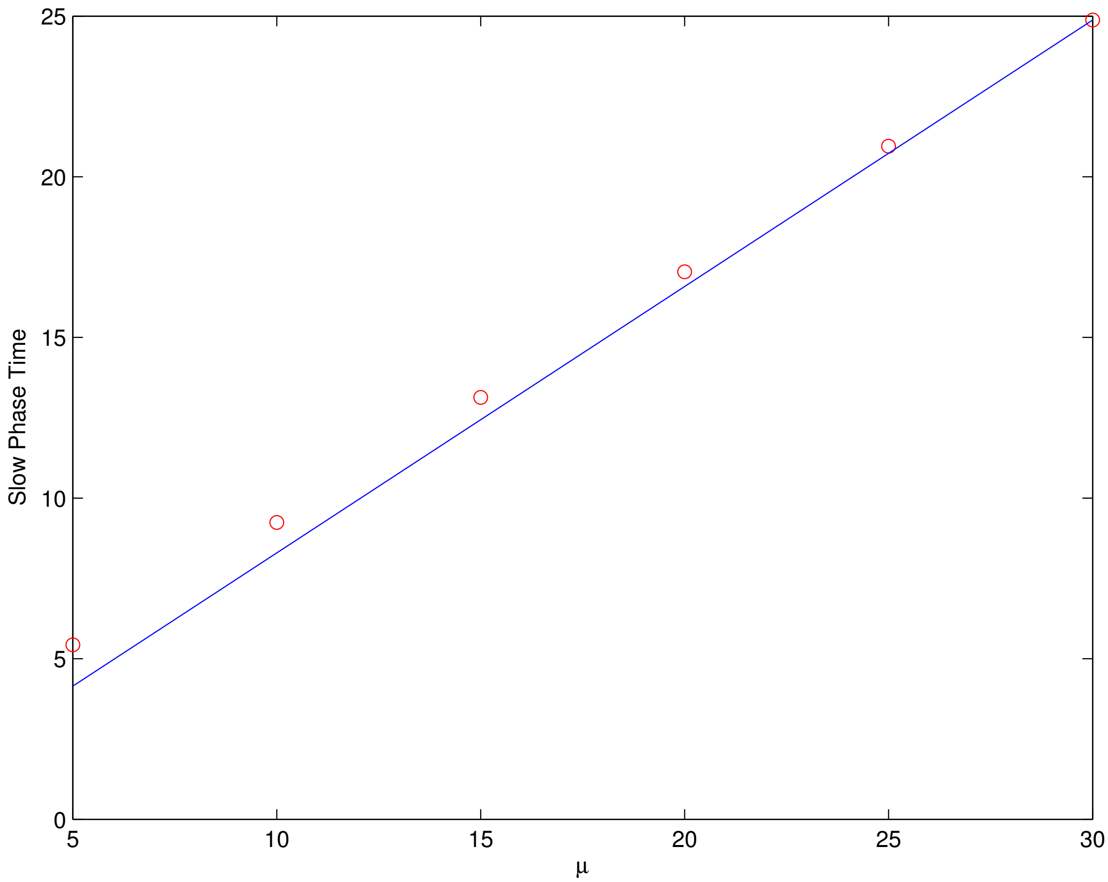
1. (a) In the weakly nonlinear regime the solution slowly spirals out to nearly symmetric periodic solution with amplitude $2+O(\mu)$. In the strongly nonlinear regime the solution has a fast and a slow phase and consequently a sawtooth structure.



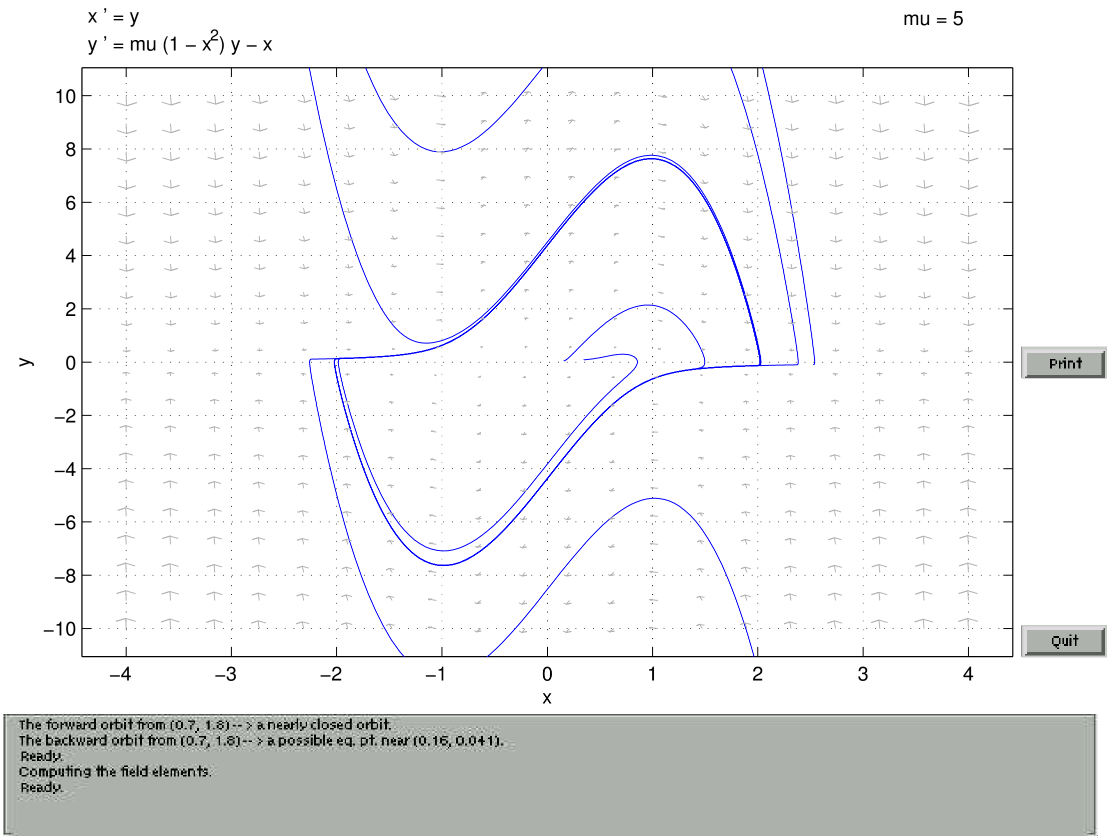


(b) In the strongly nonlinear regime we expect the slow phase to take a time of $O(\mu)$. This plot shows the time taken by the slow phase (obtained numerically) for a few values of μ. Red circles show numerical times and the blue line is the theoretical prediction that the time should be linear in μ. Notice that the actual solution deviates from the theoretical solution as μ decreases. Since very little time is spent in the fast phase, we expect the period to scale like μ.

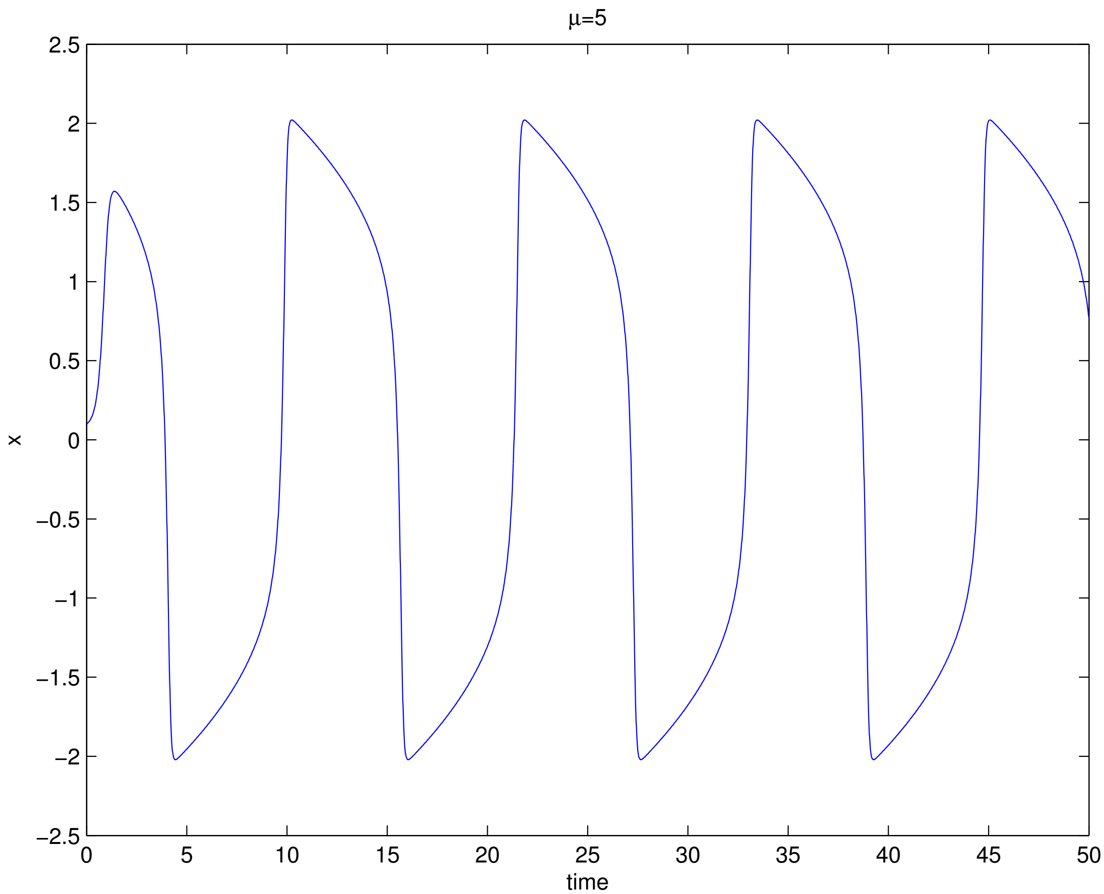
The time spent in the fast phase is strongly dependent on how the fast phase is defined, so a plot of this quantity is less useful and won't be provided here. Analytically we know that the time spent in the fast phase should scale like $O(\mu^{-1})$.



(c) Here is a plot of phase space with trajectories superimposed.



From the time series plot you can see that the slow phases are approximately 1<x<2 with $\dot{x}$ negative and -2<x<-1 with $\dot{x}$ positive.



2. (a)

$$r' = \langle r cos(\theta) sin(\theta) \rangle = 0 \quad (1)$$
$$r\phi' = \langle r cos^2(\theta) \rangle = \frac{1}{2} r \quad (2)$$

So our solution can be written $x_0 = r_0 cos((1+\frac{\varepsilon}{2})t + \phi_0)$. The limit cycle has amplitude r_0 and frequency $\omega = 1 + \frac{\varepsilon}{2} + O(\varepsilon^2)$. Applying IC: x_0=a $cos((1+\frac{\varepsilon}{2})t)$. The exact solution is x=a cos($\sqrt{1+\varepsilon t}$), confirming our perturbation solution.

(b)

$$r' = -\langle r^2 cos(\theta) sin^2(\theta) \rangle = 0 \quad (3)$$
$$r\phi' = -\langle r^2 cos^2(\theta) sin(\theta) \rangle = 0 \quad (4)$$

The amplitude and the phase are constant to O(ε^2). x_0=a cos(t).

(c)

$$r' = -r^5 \langle cos^2(\theta) sin^4(\theta) \rangle + r^3 \langle sin^4(\theta) \rangle = -\frac{1}{16} r^5 + \frac{3}{8} r^3 \quad (5)$$
$$r\phi' = -r^5 \langle cos^3(\theta) sin^3(\theta) \rangle + r^3 \langle sin^3(\theta) cos(\theta) \rangle = 0 \quad (6)$$

So a limit cycle with $r = \sqrt{6}$ and $\omega = 1 + O(\varepsilon^2)$ is approached in the long term.

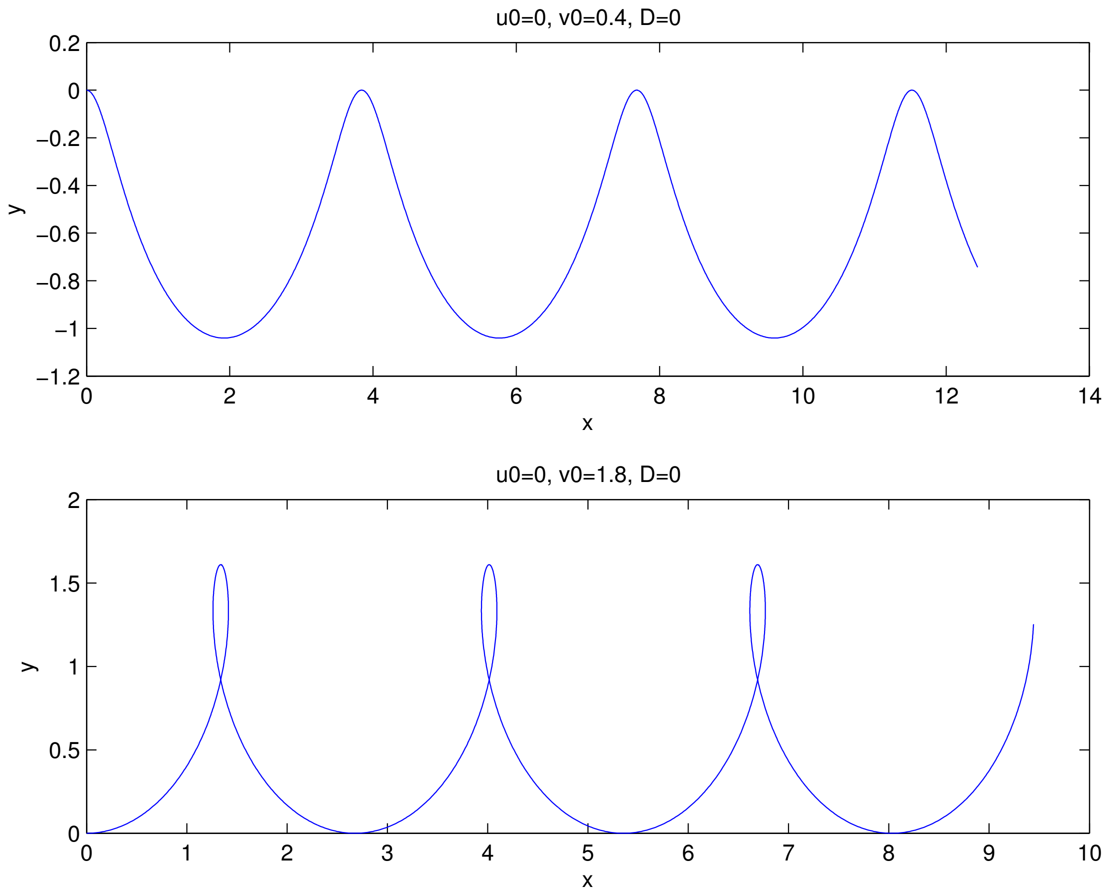
3. (a) E=$\nu^3 - 3\nu cos(u)$, so:

$$\begin{aligned} \dot{E} &= 3\nu^2\dot{\nu} - 3cos(u)\dot{\nu} + 3\nu sin(u)\dot{u} && (7) \\ &= (3\nu^2 - 3cos(u))(-sin(u)) + 3sin(u)(-cos(u) + \nu^2) && (8) \\ &= 0 && (9) \end{aligned}$$

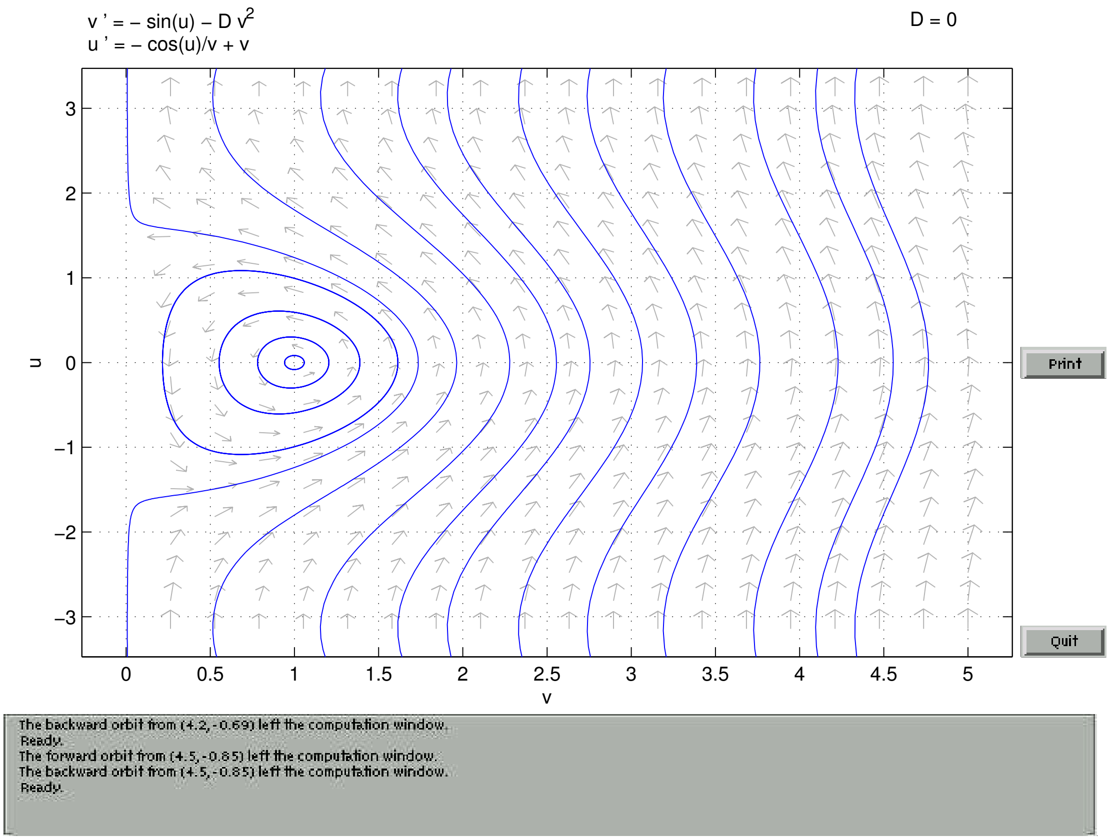
Since E must be conserved along trajectories $\nu(u)$ is specified by the equation $E = \nu(u)^3 - 3\nu(u)cos(u)$. E takes a different value for each trajectory.

(b) Orbits with E<0 can never have cos(u)<0 (so for E<0 $-\frac{\pi}{2} < u < \frac{\pi}{2}$) while orbits with E>0 can have u take any value. So we expect the separatrix to occur at E=0. Calling $\tilde{\nu}(u)$ the equation for the separatrix we have, $\tilde{\nu}(u) = \sqrt{3cos(u)}$ (we only consider non-negative ν here).

(c) The upper panel is inside the separatrix while the lower panel is outside it.



Trajectories inside the separatrix circle the fixed point while those outside it wind around the cylinder.

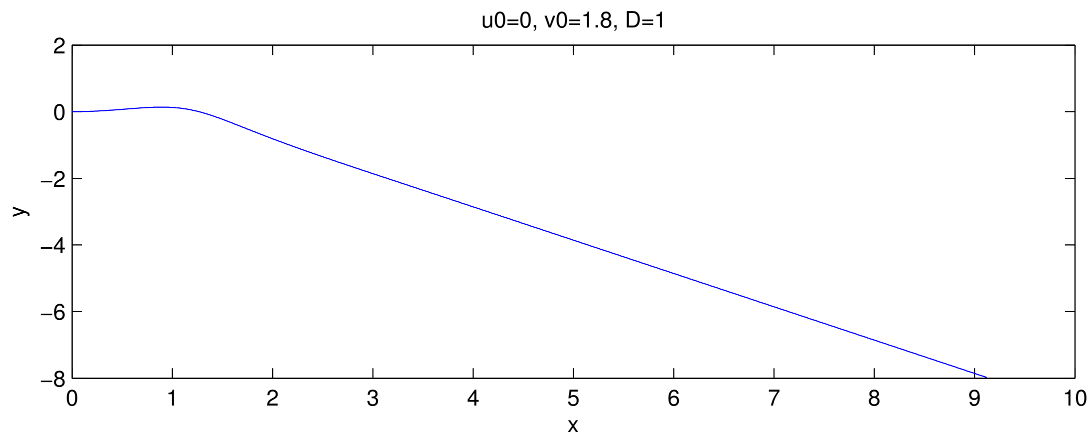


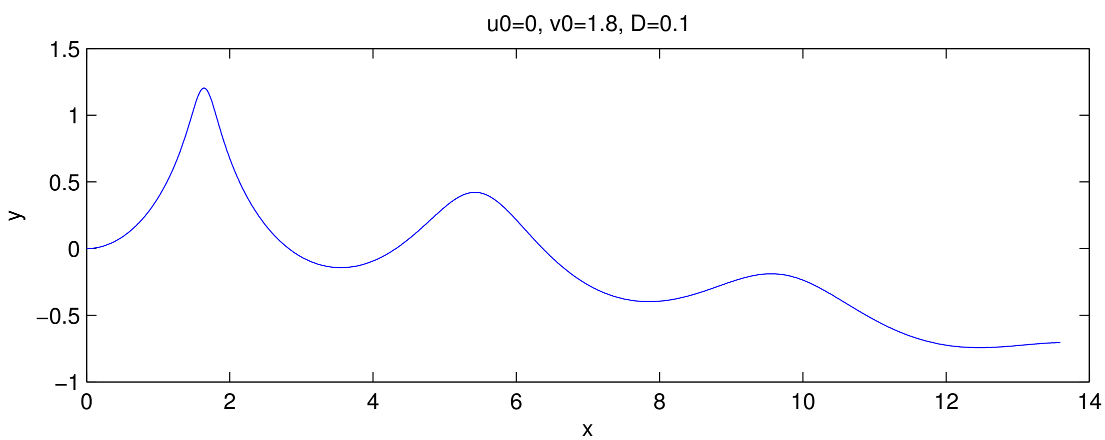
(d) The f.p. of the system is at u=-arcsin($\frac{D}{\sqrt{1+D^2}}$), v=$(\frac{1}{1+D^2})^{\frac{1}{4}}$. The Jacobian is:

$$J = \begin{pmatrix} -2Dv & -cos(u) \\ 1+\frac{cos(u)}{v^2} & \frac{sin(u)}{v} \end{pmatrix}$$

So $\tau = -2Dv + \frac{sin(u)}{v} = \frac{-3D}{(1+D^2)^{\frac{1}{4}}}$ and $\Delta = -2Dsin(u) + cos(u) + \frac{cos^2(u)}{v^2} = 2\sqrt{1+D^2}$ while $\tau^2 - 4\Delta = \frac{D^2-8}{\sqrt{1+D^2}}$. So the f.p. is a stable spiral for 0<D<$\sqrt{8}$ and a stable node for $\sqrt{8}$<D.

(e) Plots without friction are shown above. Here are some plots with friction. The glider eventually lands when there is friction.





4. (a) Rewrite the system as a 3D dynamical system:

$$\dot{w} = (\varepsilon - \alpha z)w - \omega^2 u \quad (10)$$
$$\dot{u} = w \quad (11)$$
$$\dot{z} = u^2 - \tau z \quad (12)$$

The fixed point is at the origin.

(b) Expand each variable in terms of small parameter δ to order δ^3:

$$w(t) = w_0(T_0,T_1,T_2) + \delta w_1(T_0,T_1,T_2) + \delta^2 w_2(T_0,T_1,T_2) + \delta^3 w_3(T_0,T_1,T_2) + O(\delta^4) \quad (13)$$
$$u(t) = u_0(T_0,T_1,T_2) + \delta u_1(T_0,T_1,T_2) + \delta^2 u_2(T_0,T_1,T_2) + \delta^3 u_3(T_0,T_1,T_2) + O(\delta^4) \quad (14)$$
$$z(t) = z_0(T_0,T_1,T_2) + \delta z_1(T_0,T_1,T_2) + \delta^2 z_2(T_0,T_1,T_2) + \delta^3 z_3(T_0,T_1,T_2) + O(\delta^4) \quad (15)$$

Here $T_0 = t$, $T_1 = \delta t$, and $T_2 = \delta^2 t$. If we set ε=0 the system approaches the origin, so the zeroth order terms should be zero. Another way to think about this is that the zeroth order behavior is for the system to be near the fixed point (at the origin) we expect no zeroth order terms. If u and w are O(δ), we expect z to be O(δ^2) so ε will affect the solution at O(δ^2). Define $\varepsilon = \tilde{\varepsilon}\delta^2$ where $\tilde{\varepsilon}$ is O(1).

(c) Expand to O(δ):

$$\frac{\partial w_1}{\partial T_0} + \omega^2 u_1 = 0 \quad (16)$$
$$\frac{\partial u_1}{\partial T_0} = w_1 \quad (17)$$
$$\frac{\partial z_1}{\partial T_0} + \tau z_1 = 0 \quad (18)$$

Which has the solution:

$$u_1 = A(T_1,T_2)cos[\omega T_0 + \beta(T_1,T_2)] \quad (19)$$
$$w_1 = -A(T_1,T_2)\omega\, sin[\omega T_0 + \beta(T_1,T_2)] \quad (20)$$
$$z_1 = \Delta(T_1,T_2)exp(-\tau T_0) \quad (21)$$

Since T_0 and τ are O(1), $z_1 \to 0$ on a time scale much faster than variations in T_1 and T_2, so we will take $z_1 = 0$ in what follows. This is consistent with the scaling analysis above.

Expanding to O(δ^2) and using $z_1 = 0$:

$$\frac{\partial w_2}{\partial T_0} + \omega^2 u_2 = -\frac{\partial w_1}{\partial T_1} - \alpha z_1 w_1 = \frac{\partial A}{\partial T_1}\omega sin[\omega T_0 + \beta] + A\omega\frac{\partial \beta}{\partial T_1}cos[\omega T_0 + \beta] \quad (22)$$
$$\frac{\partial u_2}{\partial T_0} - w_2 = -\frac{\partial u_1}{\partial T_1} = -\frac{\partial A}{\partial T_1}cos[\omega T_0 + \beta] + A\frac{\partial \beta}{\partial T_1}sin[\omega T_0 + \beta] \quad (23)$$
$$\frac{\partial z_2}{\partial T_0} + \tau z_2 = u_1^2 - \frac{\partial z_1}{\partial T_1} = A^2 cos^2[\omega T_0 + \beta] \quad (24)$$

To avoid singular terms, we must take $\frac{\partial A}{\partial T_1} = \frac{\partial \beta}{\partial T_1} = 0$. And we are left with:

$$u_2 = C(T_1,T_2)cos[\omega T_0 + D(T_1,T_2)] \quad (25)$$
$$w_2 = -C(T_1,T_2)\omega\, sin[\omega T_0 + D(T_1,T_2)] \quad (26)$$

Integrating the equation for z_2 we find (I did this in Mathematica, but it can be done by hand):

$$z_2 = \gamma(T_1, T_2) exp(-\tau T_0) + \frac{A^2[\tau^2 + 4\omega^2 + \tau^2 cos[2(\omega T_0 + \beta)] + 2\tau\omega sin[2(\omega T_0 + \beta)]]}{2\tau(\tau^2 + 4\omega^2)} \tag{27}$$

By the same argument as above the homogenous part of this solution can be neglected and we are left only with the particular part.

Now expand to O(δ^3):

$$\frac{\partial w_3}{\partial T_0} + \omega^2 u_3 = (\tilde{\varepsilon} - \alpha z_2) w_1 - \alpha z_1 w_2 - \frac{\partial w_1}{\partial T_2} - \frac{\partial w_2}{\partial T_1} \tag{28}$$

$$\frac{\partial u_3}{\partial T_0} - w_3 = -\frac{\partial u_1}{\partial T_2} - \frac{\partial u_2}{\partial T_1} \tag{29}$$

$$\frac{\partial z_3}{\partial T_0} + \tau z_3 = 2u_1 u_2 - \frac{\partial z_1}{\partial T_2} - \frac{\partial z_2}{\partial T_1} \tag{30}$$

Eliminating u_3 from the first two equations and again taking z_1=0:

$$\frac{\partial^2 w_3}{\partial T_0^2} + \omega^2 w_3 = (\tilde{\varepsilon} - \alpha z_2)\frac{\partial w_1}{\partial T_0} - \alpha w_1 \frac{\partial z_2}{\partial T_0} - \frac{\partial^2 w_1}{\partial T_2 \partial T_0} - \frac{\partial^2 w_2}{\partial T_1 \partial T_0} + \omega^2 \frac{\partial u_1}{\partial T_2} + \omega^2 \frac{\partial u_2}{\partial T_1} \tag{31}$$

At this point there is a considerable amount of algebra. If we had written our periodic solutions in terms of exponentials the algebra would have been slightly easier, but I sometimes find it easier to think in terms of sines and cosines. In any case, it is easy to see that $\frac{\partial C}{\partial T_1} = \frac{\partial D}{\partial T_1} = 0$. Using trigonometric identities such as cos(x)cos(3x)=$\frac{1}{2}$cos(x)+$\frac{1}{2}$cos(3x) we find that to eliminate the coefficient of cos$[\omega T_0 + \beta(T_1, T_2)]$ we must require:

$$\frac{\partial A}{\partial T_2} = \frac{1}{2} A\tilde{\varepsilon} - \frac{\alpha(\tau^2 + 8\omega^2)}{8\tau(\tau^2 + 4\omega^2)} A^3 \tag{32}$$

While to eliminate the coefficient of sin$[\omega T_0 + \beta(T_1, T_2)]$ we must require:

$$\frac{\partial \beta}{\partial T_2} = -\frac{\alpha\omega}{4(\tau^2 + 4\omega^2)} A^2 \tag{33}$$

Next use the $\frac{\partial A}{\partial T_2} = \frac{1}{\delta^2}\frac{\partial A}{\partial t}$ and $\frac{\partial \beta}{\partial T_2} = \frac{1}{\delta^2}\frac{\partial \beta}{\partial t}$ since A and β are independent of T_0 and T_1. Also define $a \equiv \delta A = O(\sqrt{\varepsilon})$ to find:

$$u \approx acos(\omega t + \beta) \tag{34}$$

$$\dot{a} = \frac{1}{2} a\varepsilon - \frac{\alpha(\tau^2 + 8\omega^2)}{8\tau(\tau^2 + 4\omega^2)} a^3 \tag{35}$$

$$\dot{\beta} = -\frac{\alpha\omega}{4(\tau^2 + 4\omega^2)} a^2 \tag{36}$$

These equations describe a supercritical Hopf bifurcation with ε as the control parameter. The multiple scale approximation is reasonable here because we have a solution that slowly spirals toward a limit cycle while completing many circuits of the origin.