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APM147
10/14/04

Problem Set 3 Solutions

1. Start with the equation:

$$\dot{X} = RX - X^2 + a_n X^n + O(X^{n+1}) \tag{1}$$

Consider the near identity transformation:

$$x = X + b_n X^n + O(X^{n+1}) \tag{2}$$

Invert this transformation:

$$\begin{aligned} X &= x - b_n X^n + O(X^{n+1}) \\ &= x - b_n(x - b_n X^n + O(X^{n+1}))^n + O(X^{n+1}) \\ &= x - b_n x^n + O(x^{n+1}) \end{aligned} \tag{3}$$

Differentiate equation (2):

$$\dot{x} = \dot{X} + b_n n X^{n-1}\dot{X} + O(X^n)\dot{X} \tag{4}$$

Now use equation (1) and the inverted near identity transformation:

$$\begin{aligned} \dot{x} &= (RX - X^2 + a_n X^n) + b_n n X^{n-1}(RX - X^2 + a_n X^n) + O(X^{n+1}) \\ &= RX - X^2 + (a_n + Rnb_n)X^n + O(X^{n+1}) \\ &= R(x - b_n x^n) - (x - b_n x^n)^2 + (a_n + Rnb_n)(x - b_n x^n)^n + O(x^{n+1}) \\ &= Rx - x^2 + (a_n + R(n-1)b_n)x^n + O(x^{n+1}) \end{aligned} \tag{5}$$

So the x^n term disappears if we choose $b_n = -\frac{a_n}{R(n-1)}$ and we are left with a system that is closer to the normal form.

2. (a) $\dot{\theta} = \mu sin\theta - sin2\theta$

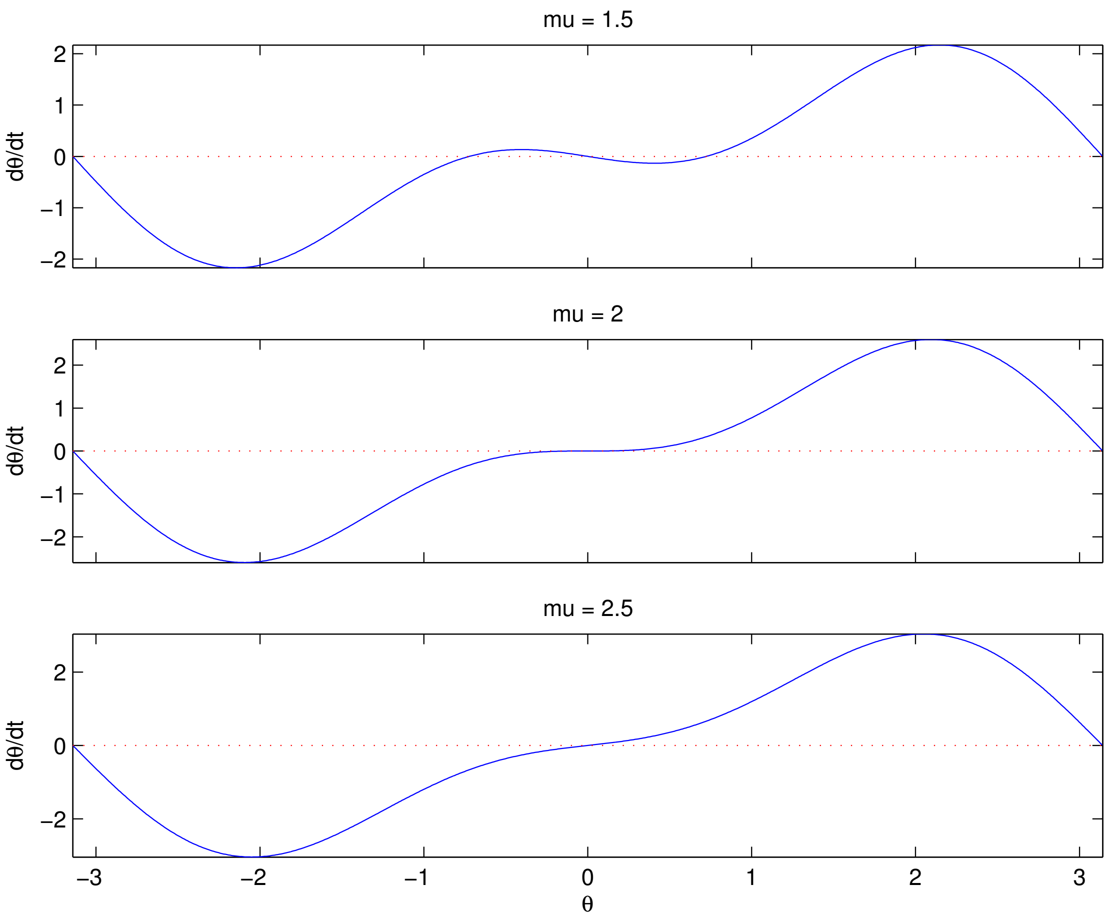

Figure 1: A subcritical pitchfork bifurcation occurs at μ=2, θ=0.

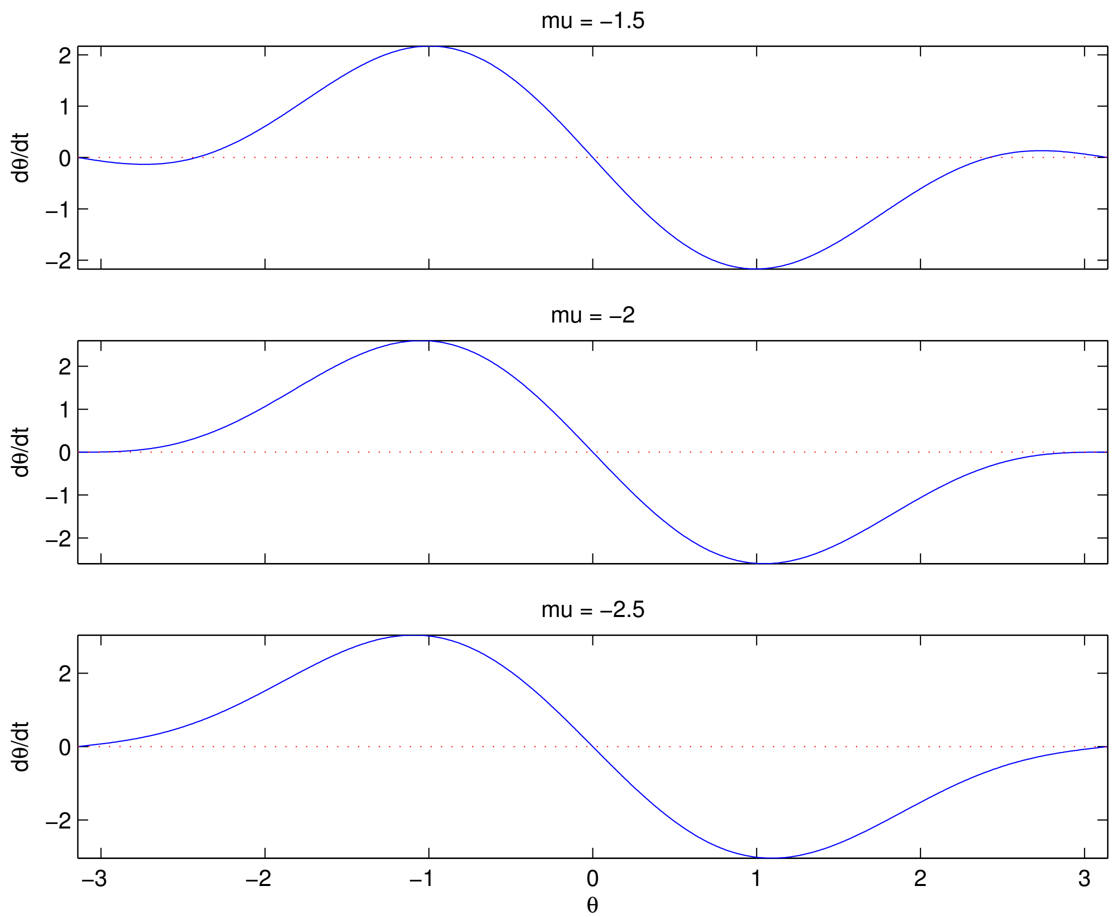

Figure 2: A subcritical pitchfork bifurcation occurs at μ=-2, θ=π.

(b) $\dot{\theta} = \mu + cos\theta + cos2\theta$

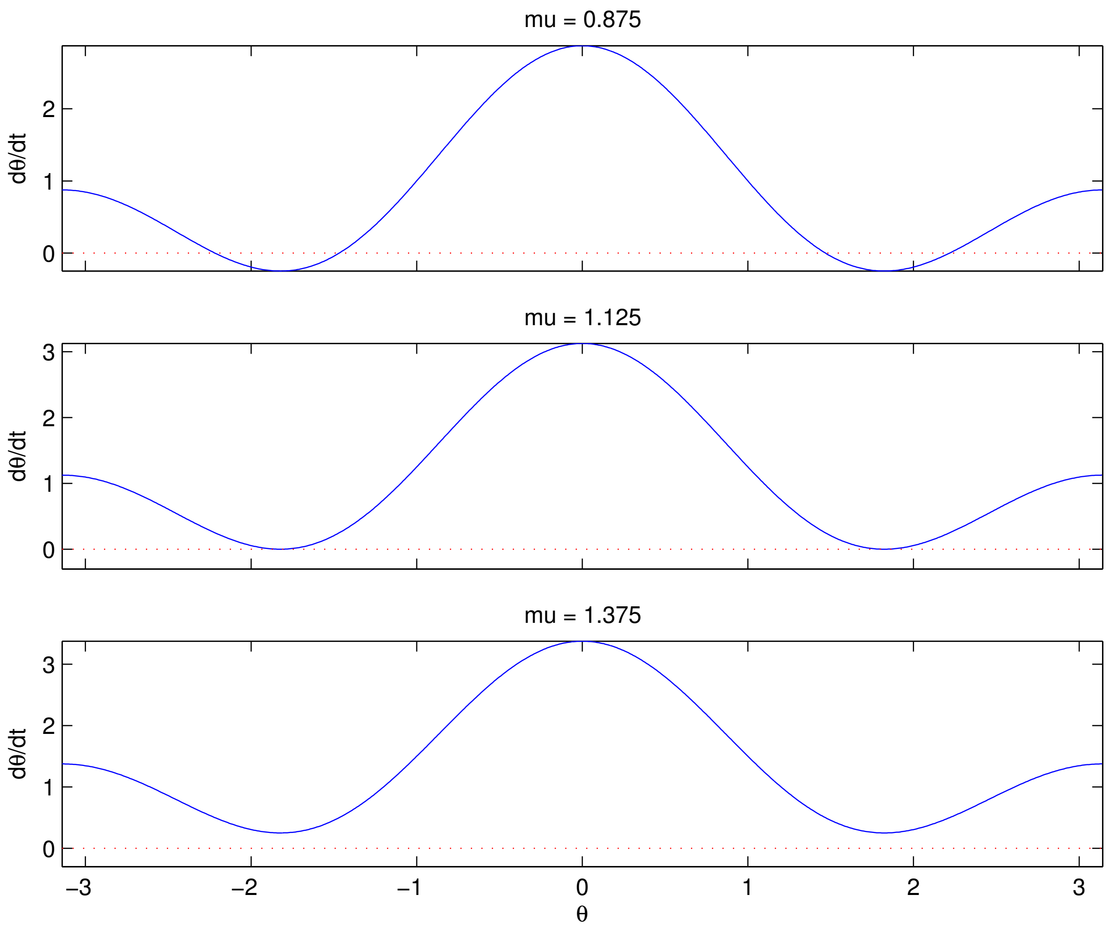

Figure 3: Saddle-node bifurcations occur at $\theta = \pm arccos(\frac{-1}{4})$, $\mu = \frac{9}{8}$.

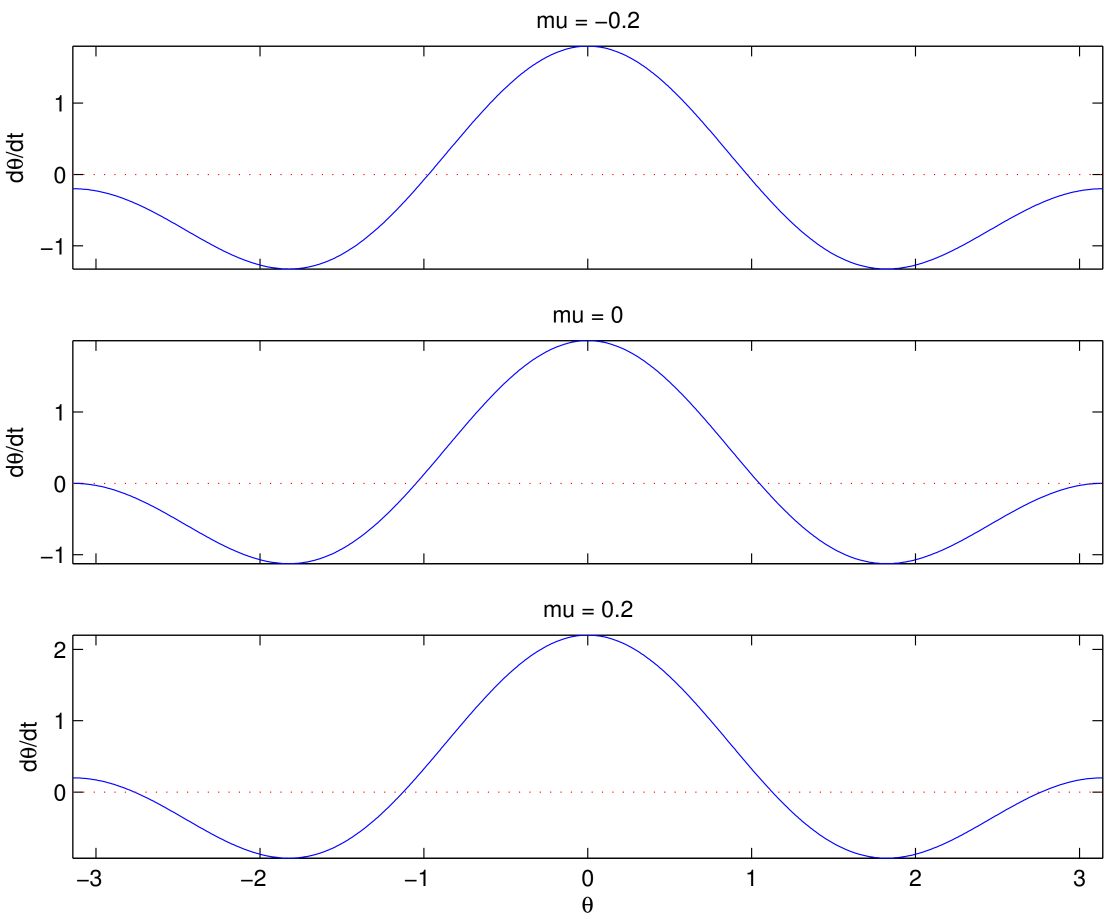

Figure 4: A saddle-node bifurcation occurs at μ=0, θ=π.

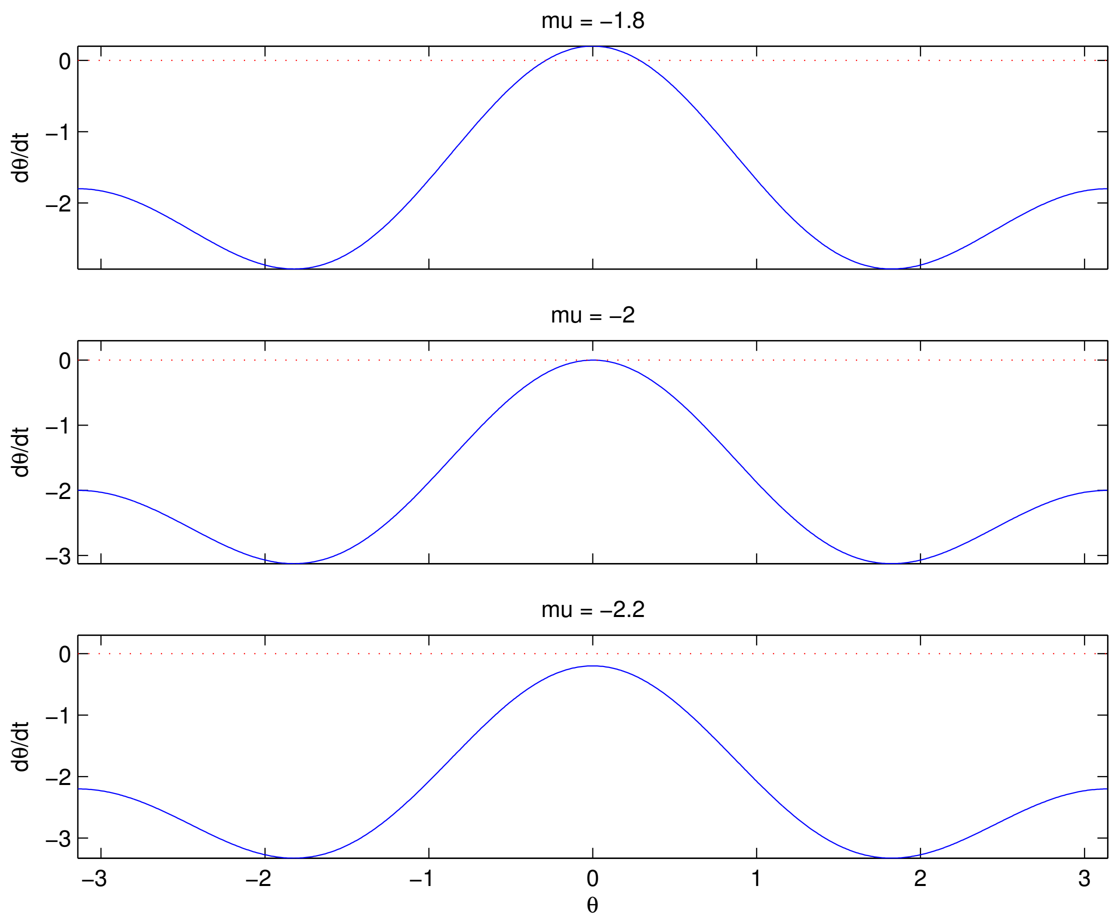

Figure 5: A saddle-node bifurcation occurs at μ=-2, θ=0.

(c) $\dot{\theta} = \mu + sin\theta + cos2\theta$

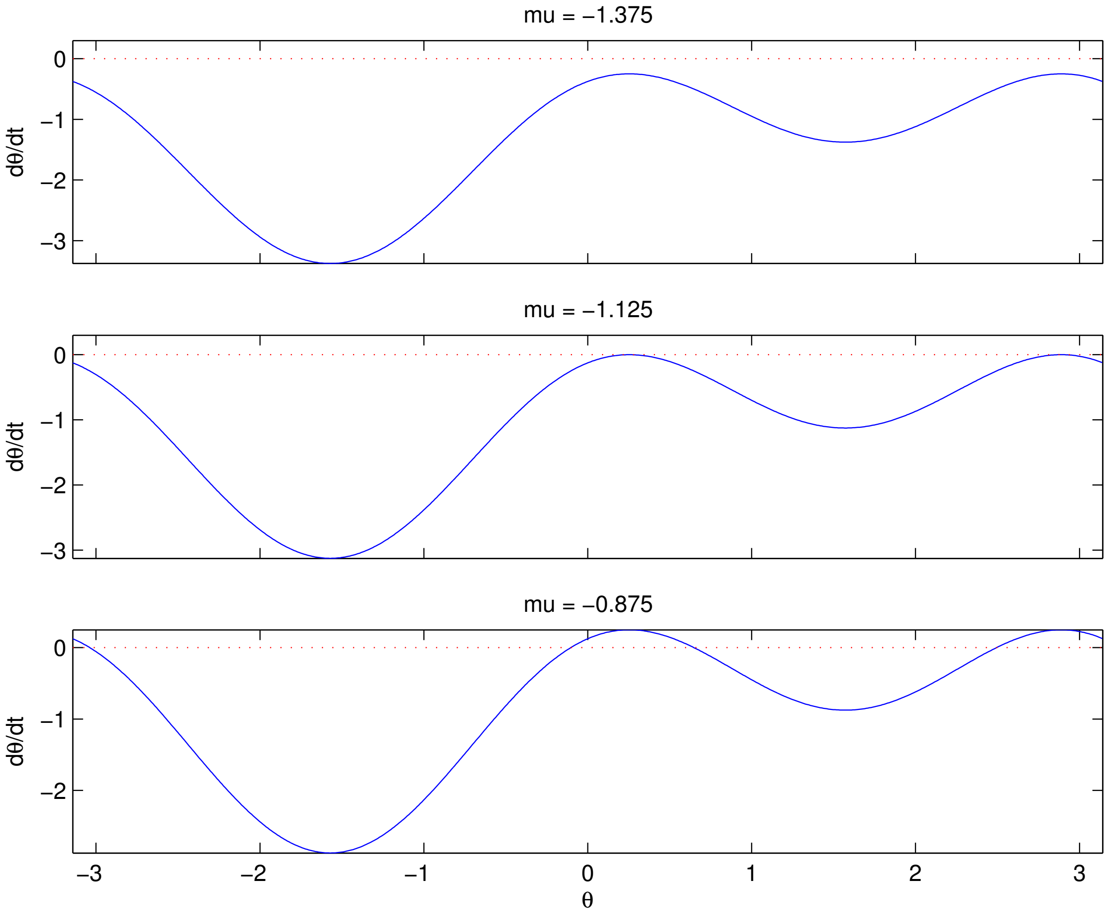

Figure 6: Saddle-node bifurcations occur at $\mu = \frac{-9}{8}$, $\theta = arcsin(\frac{1}{4}), \pi - arcsin(\frac{1}{4})$.

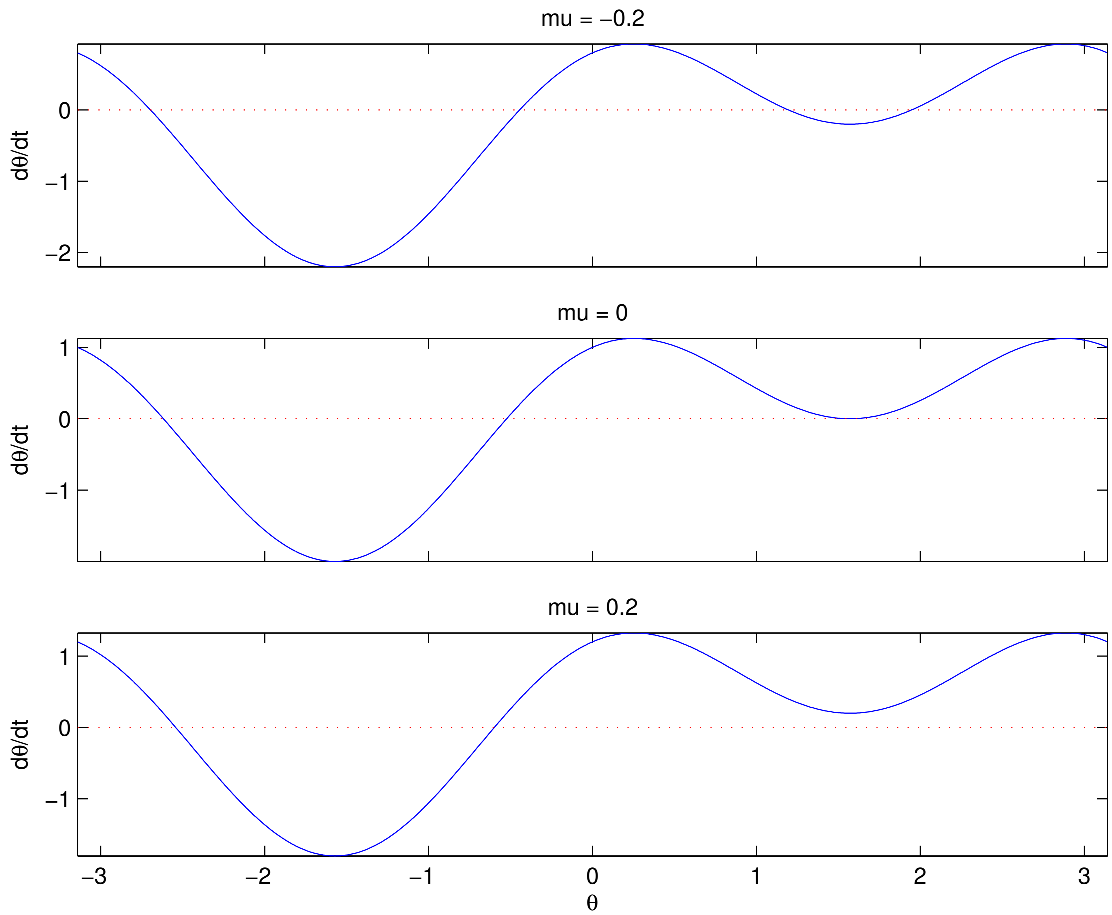

Figure 7: A saddle-node bifurcation occurs at μ=0, $\theta = \frac{\pi}{2}$.

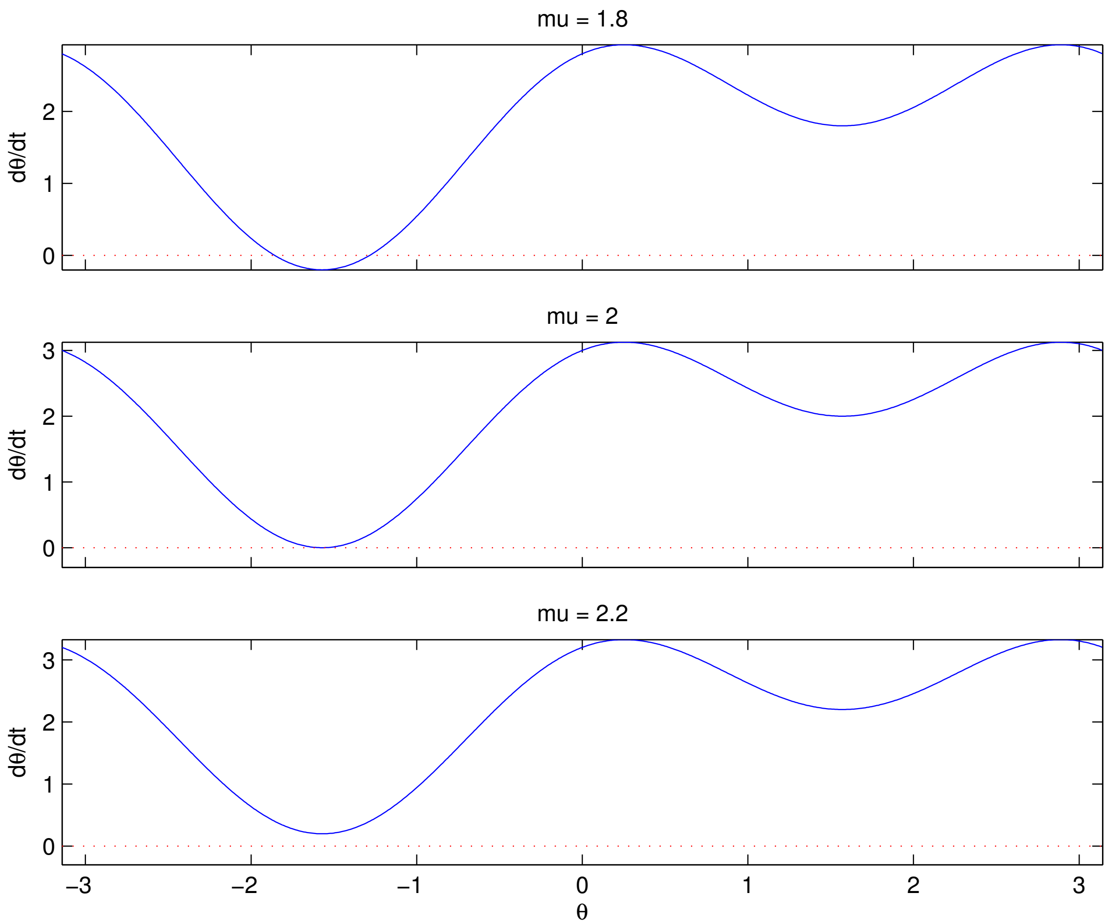

Figure 8: A saddle-node bifurcation occurs at μ=2, $\theta = \frac{-\pi}{2}$.

(d) $\dot{\theta} = \frac{sin 2\theta}{1+\mu sin\theta}$

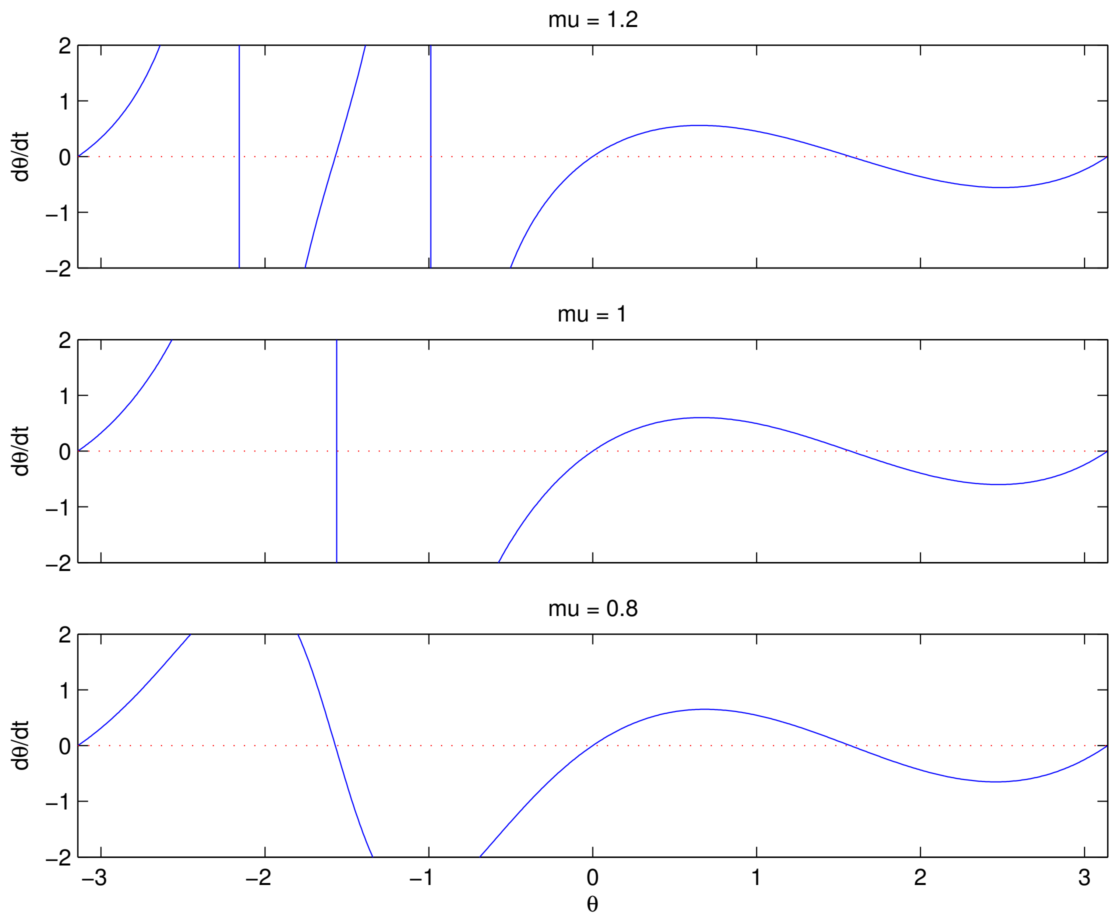

Figure 9: At μ=1, the fixed point at $\theta = \frac{-\pi}{2}$ switches stability. This also happens at μ=-1 and $\theta = \frac{\pi}{2}$. The meaning of this is somewhat questionable since the system becomes unphysical at these points.

3. The equation is integrated for a number of different initial conditions. Notice that the nearest stable fixed point is approached in each case.

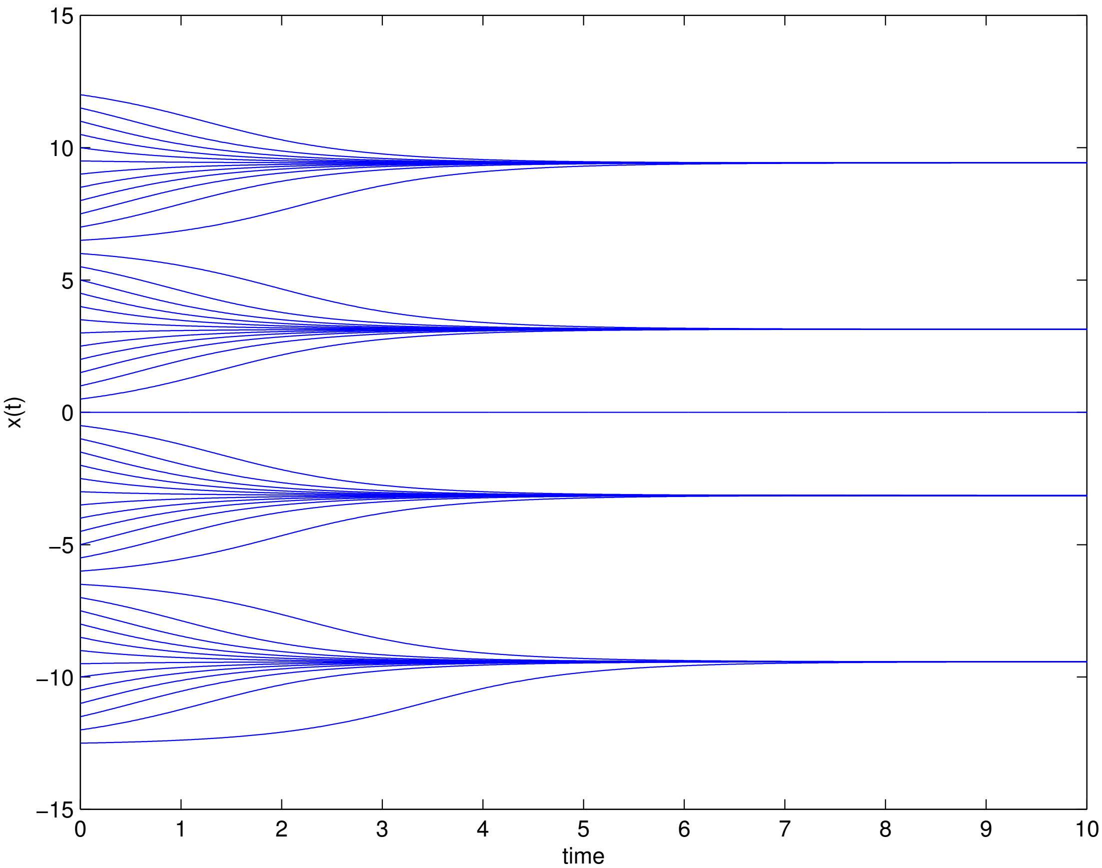

Figure 10: Integrating with modified Euler scheme.

4. For δ=0, the equation becomes:

$$\dot{x} = -\mu x + x^3 \tag{6}$$

So a subcritical pitchfork bifurcation occurs at x=0, μ=0. If we look near x=0 when δ is nonzero, we have:

$$\dot{x} \approx -\mu x + \delta x^2 \tag{7}$$

For nonzero δ, a transcritical bifurcation happens at x=0, μ=0. It doesn't matter how small δ is, as long as it is nonzero, this will be a transcritical, not a pitchfork bifurcation.

Let's solve explicitly for x^*, our fixed points, by setting $\dot{x}$=0. We get:

$$x^* = 0, -\frac{\delta}{2} \pm \sqrt{(\frac{\delta}{2})^2 + \mu} \tag{8}$$

It is now clear that saddle-node bifurcations occur along the line $\mu = -(\frac{\delta}{2})^2$, $\delta \neq 0$, at the point x=$-\frac{\delta}{2}$.

Here are some plots:

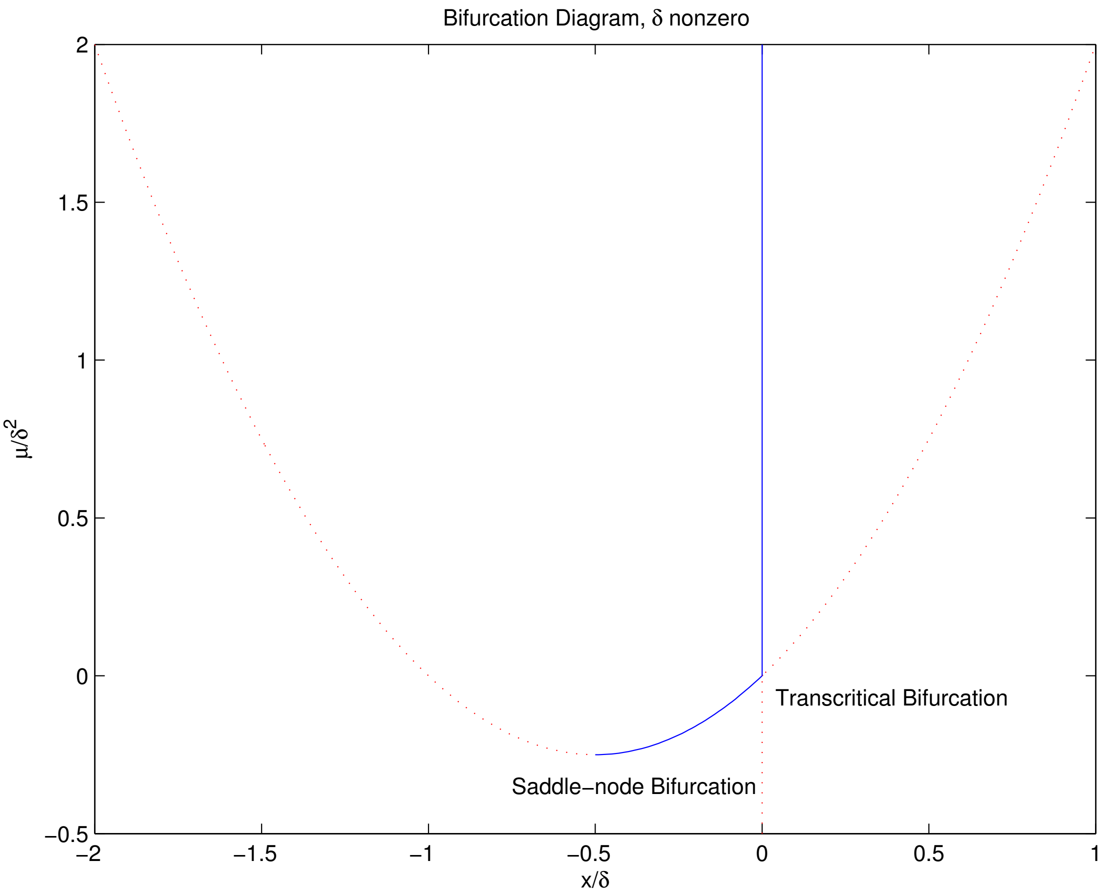

Figure 11: Bifurcation diagram

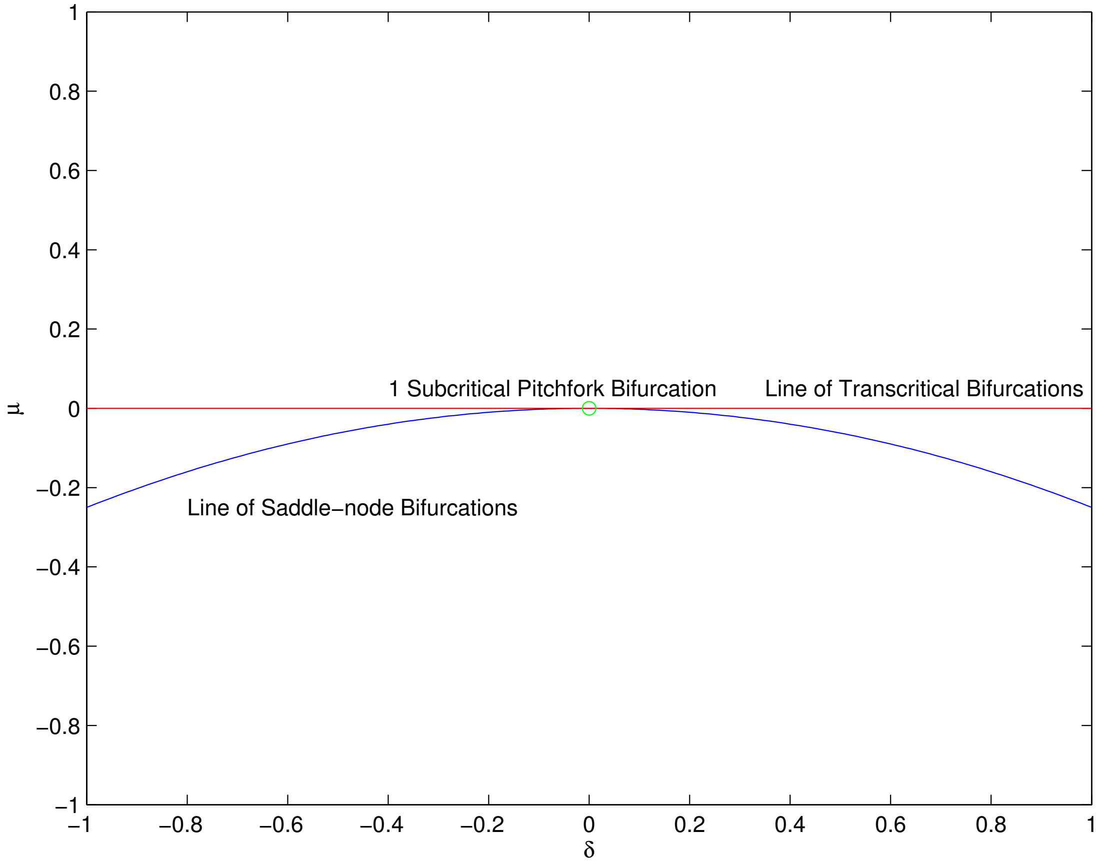

Figure 12: Investigating δ-μ space