Mathematical modeling

Applied Mathematics 115
(Spring 2008)

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Workshops: (sections): first meetings will be Wednesday Feb 6 on 2-4 and 4-6; following weeks: on Mon 4-6pm and Wed 4-6pm; location: SEAS computer lab.

Day & time: Tue-Thu 11:30-1pm.

Location: Maxwell Dvorkin 125;

1st meeting: Thursday Jan 31st.

Bibliography: material from several textbooks and other sources will be used; the source materials for all lectures, including Matlab programs used in class, may be found here. Follow links below for the specific source material for each lecture.

Web: http://my.harvard.edu/icb/icb.do?keyword=k25683

- Advice on projects and presentations

This document: http://www.seas.harvard.edu/climate/eli/Courses/APM115/2008spring/detailed-syllabus-apm115_modelling.pdf

Announcements Last updated: April 24, 2008.
Feel free to write call or visit us with any questions.

Workshops:

Projects:

1 Outline

Mathematical models are ubiquitous, providing a quantitative framework for understanding, prediction and decision making in nearly every aspect of life, ranging from timing traffic lights, to controlling the spread of disease, to weather, climate or earth quakes, to economic forecasting. This course provides an introduction to modeling through in depth discussion of a series of real examples.

2 Administrative

Prerequisites: Applied Mathematics 21a and 21b, or Mathematics 21a and 21b or permission of instructor. Knowledge of some programing language is helpful but not necessary as we'll be introducing Matlab for those with no previous experience.

Workshop: In addition to the lectures, the class will have a weekly computer workshop, in which students will have the opportunity to implement and explore the models described in class on a computer. The workshops should be entirely self contained, the goal being that every student will have successfully implemented the model by the end of the workshop. Occasionally some groups may be asked to complete the task after the workshop. Regular times for workshops will be scheduled at the beginning of the semester.

Computer Skills: No computer skills are assumed for this class. Computer labs will be run in MATLAB, and as part of the course, students will therefore gain facility with this package. Students should download and install the MATLAB system on their computers from the FAS software site (see the WWW links button on the main course page). We will schedule a crash introduction to MATLAB early next week. Additional instruction will be given in the computer workshops.

Group homework: Homework assignments, in the form of model development projects, will be assigned roughly every two weeks. These projects will be completed by 3 person teams (to be formed in the first workshop). These models will further explore the models discussed in class. Each team will focus on a different extension of the models, and will present their findings in a subsequent class. A brief written report is expected from each group for each of the projects.

Individual final project: During roughly the last third of the semester, each individual will carry out a final project investigating a new model or carrying out a significant extension of a model discussed in class. Student creativity in selecting and refining the project topic will be important in evaluation of the project. During this time period, homework will not be assigned. Class time will be spent discussing "progress reports" on the projects.

Grading: Attendance in lectures and workshops is mandatory and will be part of the final grade; class Participation will also be an important consideration. Workshops 20%; Homework (student group projects) 30%; Final Project 50%;

3 An evolving syllabus

This list of topics to be covered will evolve and become more specific and detailed during the course. Follow links to see the source material and Matlab demo programs used for each lecture. We'll be building a toolbox of modeling concepts and methods during the course, and the list below indicates the toolbox items provided by each of the topics.

L1: Introduction, overview here.

L2-3: POPULATION DYNAMICS, SINGLE SPECIES: [**toolbox concepts:** ODEs; fixed points, linear stability of fixed points, numerical solution of ODEs, Markov process, stochastic equations, stochastic simulation, equations for probability distribution function]. downloads

- *A single species with limited resources, deterministic approach:* logistic equation, geometric approach, linearized stability analysis (Strogatz, 1994, 2.3, 2.4 p 21-25, here).
- *A single species, stochastic approach:* probability, Markov process, simulating stochastic process and calculating the pdf numerically, (Renshaw, 1991, 3.2.3, pp 55-57; 3.4, p 59-61, here); master equation, solving directly for the pdf (3.1, p 46-48);
- stochastic_logistic.m.

L4-5: POPULATION DYNAMICS, COMPETITION OF SPECIES: [**toolbox concepts:** phase plane, oscillations vs limit cycles, model validation; maps as dynamical systems, their fixed points, oscillations and chaos] downloads

- *Two competing species:* deadly survival struggle between sheep and Rabbits (Strogatz, 1994, 6.4, p 155-159), or parts of pp 1, 2, 5 in notes. Sheep and rabbits equations, intro to phase plane, fixed points, stability, limit cycles. Linearization in 2d, quiver_sheep_rabbits.m.
- *Predator-prey oscillations:* Motivation: Lynx and snowshoe hare in Canada. Lotka-Volterra equations (eqn 4.72, Mesterton-Gibbons, 1988, , section 4.12, p 154-157, here), (or from wikipedia) and an improvement resulting in a limit cycle (10.2, p 389-393), predator_prey.m. An interesting twist: model validation using Lynx and snowshoe hare observations.
- *Logistic map:* fixed points, stability, oscillations, chaos. Why is the logistic map behavior so much richer than of the continuous logistic equation, Poincare section and the relation between maps and continuous systems. (Strogatz, 1994, 10.0-10.4, p 348-363, here).
- *Examples of projects:* bugs, birds and leaves from Strogatz; love affairs;

- [Assignments of first group projects]

L6-7: CLIMATE: [**toolbox concepts:** multiple equilibria, bifurcations, hysteresis, catastrophes, stochastic differential equations, white noise, red noise, Fourier transform, spectrum, AR(n) Markov process, variance, autocorrelation] downloads

- *Energy balance models, from greenhouse warming to snowball earth:* Background: black body radiation, solar radiation, long wave radiation, albedo, greenhouse effect, emissivity, schematic energy budget for the atmosphere. A zero-dimensional energy balance model including the ice albedo feedback; results: saddle node bifurcation, catastrophes and hysteresis due to two back to back saddle node bifurcations (Aarnout van Delden's slides 1-9, saddle-node bifurcation, lower half of page 47 in notes). Implications: snowball earth, letter to the president; energy_balance_0d.m.

- *A stochastic model of Climate variability:* Motivation: weather vs climate, fast vs slow variability; spectrum, white noise, red noise, Hasselmann's stochastic model, simulating and then solving exactly in Fourier space, solving for the climate temperature spectrum, (notes) Hasselmann.m; an equivalent discrete Markov process (AR(1)), variance and autocorrelation (notes). ar1.m.
- *Thermohaline circulation and the day after tomorrow:* Motivation: glacial cycles, Dansgaard Oeschger events, thermohaline circulation, day after tomorrow, trailer. Stommel's model, multiple equilibria (bi-stability) of ocean circulation, and abrupt climate change (Aarnout van Delden's slides 1-3, 8-11, about the Stommel-Taylor model). Stommel.m.
- *Examples of projects:* room air conditioning; 1d energy balance model;

L8: MODELING "PHILOSOPHY": Why model? What's a good model? (as simple as possible, equations well motivated by data, aids our understanding, and able to surprise us and provide new insight/ prediction not obvious beforehand). Model validation, simple vs complex models, simulation vs modeling, optimization, stochastic vs deterministic, etc. notes.

L9-10 GENETICS [**toolbox concepts:** Markov chains, stochastic processes, stochastic matrices.] (Roberts (1976), chapter 5, see also Olinick (1978) chapter 10) downloads

- *Background: stochastic processes and Markov chains:* Stochastic processes with and without memory, Markov chains (5.1, examples 2, 3). Transition probabilities and transition digraphs (5.2, examples 2,3,11 (coin, Russian roulette, gambler's ruin); theorems 5.1, 5.2). Classification of states and chains: strongly connected set, closed sets, ergodic sets, absorbing states (5.3); absorbing chains, expected time to absorption and the fundamental matrix for an absorbing chain (theorem 5.6); stationary probability, Perron-Frobenius theorem (wikipedia), bounds on the speed of convergence using second-largest eigenvalue.
- *Genetics:* (5.7) genes: recessive, dominant; states: pure dominant, pure recessive, hybrid; three scenarios: continued crossing with a hybrid; continued crossing with a dominant; inbreeding.
- *Projects:* Simulate a stochastic adjustment process, compare the results to theoretical predictions (if available) or intuitions (if not).

L11: THE INTERNET AND GOOGLE'S PAGERANK: [**toolbox concepts:** stochastic matrices, eigenvalues, power method for calculating eigenvalues/ vectors, networks] downloads

- Motivation: building a \$100 Billion company based on a simple model; Google vs BMW: this and this.
- Modeling the Internet via a random walker and the PageRank algorithm from p 1-7 here; the theoretical background, proving that there is a PageRank and that it is unique is the Perron-Frobenius Theorem given in section 6.1 of the file. See also Wikipedia for the theorem and for stochastic matrices; the power method is explained in (Burden and Faires, 2004, 9.2 p 557-558) and in wikipedia.

L12: First group project presentations

- [Assignments of second group projects]

L13-14: TRAFFIC FLOW: [**toolbox concepts:** discrete modeling, delayed ODEs, PDEs, waves, characteristics, shocks] downloads

- *Motivation:* Think your commute is bad? watch the traffic in Karachi from youtube or locally; Local copies of two 3d animations from here are: driver to helicopter perspective, and merging traffic from bird view; simple gif traffic animations; A very nice Java applet demonstrating many different traffic wave related phenomena here; report on the effects of adaptive cruise control on traffic jams;
- *Single-car approach:* deriving relation between inter-car distances and car velocity (Mesterton-Gibbons, 1988, 1.13, p 34-35, here), velocity-density relationship (2.3, p 57-58), propagation of a perturbation, driver reaction time and stability of traffic flow (2.8, p 76-83). traffic.m.
- *Macro approach:* car conservation and a kinematic wave equation (Haberman, 2003, 12.6.2, p 548-549, here), method of characteristics and graphical solution (12.6.3, p 549-550), example from my LaTeX notes, fan-like characteristics (p 551), development of discontinuities, shock waves and traffic jams (12.6.4, p 551-553), jump conditions across shock, shock velocity including example (p 554-555), finite time development of shocks and shock dynamics (p 556-557). Traffic light problem (Whitham, 1974, 3.1, p 71-72, here).

L15-16: Diffusion: downloads

notes; Derivation of the 1d and 2d diffusion equations (e.g. for heat diffusion, Adam (2003), here, from the last paragraph of page 311 to end of first paragraph on page 313). As for boundary conditions, describe two alternatives: specified temperature (e.g. $T(x=0,L;t) = T_0$) and specified flux boundary conditions (e.g. $-\kappa\frac{dT(x=0,L;t)}{dx} = 0$). Values of diffusion coefficients for various materials, and scaling/ dimensional arguments for diffusion time (beginning of page 311, and rest of page 313). Numerical solution by straight forward center finite differencing in space and the Euler time stepping scheme. Examples including the "melting" of an initial structure, a point source, and a rotating point source (all solved by diffusion_demo_apm115.m).

L17: Second group project presentations

L18: Assignment of third group project

L19-20: SPATIALLY EXTENDED DISCRETE SYSTEMS: [**toolbox concepts:** discrete modeling, cellular automata (CA), fractals, fractal dimension, percolation, phase transitions; time permitting: self organized criticality and $1/f$ behavior]
Strategy here: first teach several examples and then proceed to some fundamentals and a slightly more rigorous systematics. downloads

- (first lecture) Motivation: youtube movies of fires from downloads directory: "Morgan starts a fire" and "fighting a forest fire". Then forest_fire_percolation.m and explain in detail the algorithm of this Matlab code.

- Making the forest more dynamic: adding probabilistic tree growth and lightening forest.m (Chopard and Droz, 2005, p 31-33);
- Conway's "game of life": show rules from wikipedia, show a simulation using life.m; show and work out in detail some interesting cases from wikipedia and animations from the "wikipedia commons". Discuss fixed points, periodic behavior, gliders, guns and more;
- Diffusion limited aggregation: first with learning_stage=1 (low resolution, slow) and then 0 (high resolution, fast).
- Particle dynamics and HPP rule: show Figs 2.9, 2.10 and 2.11 from Chopard-Droz pages 39-40 and run gas2.m. The HPP rule simulation of gas particles in a two-chamber container (Chopard and Droz, 2005, 2.2.5, p 38-42), gas2.m;
- (second lecture) CA background: definition, neighborhood types, boundary conditions (Chopard and Droz, 2005, 1.3.1-1.3.3, p 12-18, here);
- Systematics: 1d CA, Wolfram rules (Matlab code: ca_1D.m; tables of all rules here and here), (ir)reversibility, Totalistic CA, four universal classes (2.1.1-2.1.2, p 21-26); class 3 and sea shells wikipedia; (see also paper by Wolfram (1984) here).
- Fractals, and fractal dimension (Strogatz, 1994, 11.0-11.4, p 398-410, here). first box dimension, then note problem with it (density of points is not considered) and introduce correlation dimension.
- More 2d forest fires in a static forest: critical (threshold) tree density, percolation and fractal clusters, universal scaling behavior near percolation threshold (Chopard and Droz, 2005, p 31-33), forest_fire_percolation.m;
- Time permitting: stochastic CA, the diffusion rule and macroscopic limit leading to the 1d diffusion equation (3.1.1 p 67-68; 3.1.3, 71-74).
- Time permitting: other examples: land slides in a sand pile, scale-invariance, $1/f$, universal exponents and behavior as near a phase transition without having to vary a parameter (hence *self-organized* criticality) (Bak et all sandpile model from wikipedia), sandpile_simple.m; rules for sand flow in an hour glass (Chopard and Droz, 2005, 2.2.6 p 42-46).
- Resources: introduction to automata in Matlab.
- *Examples of projects:* snow bumps on ski slopes; sand piles; hour glass;

L21: WAVES!: [**toolbox concepts:** wave equation]
This lecture comes after the cellular automata ones. Video of stadium wave from youtube. First approach: Stadium automata based on the simple model in the nature paper with details from the Hungarian web page and their Java animation. Second approach: a wave equation in 1d. Solution and wave basics (wavelength, period, phase velocity). Then 2d.

L22: Third group project presentations

L23-27: Final individual project assignments and progress reports.

25 *Predicting the stock market (sort of...)* [**toolbox concepts:** least square fitting, autoregressive AR(p) modeling, higher order Markov processes, geometric Brownian motion]

   – Motivation: stock/ oil prices; uncorrelated white noise random numbers, 1st order Markov process, autocorrelation (as a measure of the memory) of a time series, of a white noise, of a Markov process (least-squares-notes.pdf)
   – Geometric Brownian motion (`plot_GBM.m`; reducing the stock price prediction to a least-squares problem for a few coefficients in a Markov process (above hand written notes). Another motivation: fitting a set of data points with a low-order polynomials (BF 8.1 p 482-484). Solving least-square problems: the textbook solution which is the wrong way to go (CM 5.5 first page, and above notes).

- During reading period: final presentations of individual projects, 10 minutes per presentation.

References

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