re evaluated along (t, x^*, u^*). Define

$$\delta u = u - u^*. \qquad (6)$$

u is called the *first variation* of J and is

$$\lambda') \, \delta x + (f_u + \lambda g_u) \, \delta u \big] \, dt. \qquad (7)$$

vas derived earlier by a slightly different
re flexible. It provides the first variation
ice in functional values) even when a
ctions may be difficult to construct.
hen no modification of that policy (say to
s before, we choose λ to satisfy

$$\iota^*) + \lambda(t) g_x(t, x^*, u^*)\big], \qquad (8)$$

e zero. Then we need

$$+ \lambda g_u(t, x^*, u^*)\big] \, \delta u \, dt \le 0 \qquad (9)$$

ı of the control δu. (Recall that δJ is the
ie maximum.) Note that feasibility now
responding modified state variable termi-

ection that, if there is a feasible control u
to $x(t_1) = x_1$, then the coefficient of δu

$$\lambda g_u(t, x^*, u^*) = 0 \qquad (10)$$

* are optimal for (1)–(3), then there is a
aneously satisfy (2), (3), (8), and (10).
espect to u at each t. Note that there is
value x_1 provides the needed condition.
r obvious, since δu cannot be chosen
control must be feasible; it must drive the
ssity of (10) in the present case can be
ruction, as shown in the appendix to this
otimal but does not satisfy (10). Then we
. δu that is feasible and that improves J.
ity and completes the demonstration of

ertain regularity conditions hold; other-
A full statement of necessary conditions
h f that may be either 0 or 1. We have

implicitly assumed that λ_0 can always be chosen equal to 1; yet without regularity, it may be necessary to choose $\lambda_0 = 0$. As an example, consider

$$\max \int_0^T u \, dt$$

$$\text{subject to} \quad x' = u^2, \quad x(0) = x(T) = 0.$$

In this problem, $u = 0$, $0 \le t \le T$, is the *only* feasible control. Writing

$$H = u + \lambda u^2,$$

we have

$$H_u = 1 + 2\lambda u = 0.$$

which is not satisfied by $u = 0$. The correct version is

$$H = \lambda_0 u + \lambda u^2,$$

so

$$H_u = \lambda_0 + 2\lambda u = 0.$$

A choice of $\lambda_0 = 0$ and $u = 0$ does satisfy this condition. We shall implicitly assume in the following that we can optimally choose $\lambda_0 = 1$. See also Section 14 for a more complete treatment.

Example 1. We solve our production planning problem by optimal control. Let $u(t)$ be the production rate and $x(t)$, the inventory level:

$$\min \int_0^T (c_1 u^2 + c_2 x) \, dt \qquad (11)$$

$$\text{subject to} \quad x'(t) = u(t), \quad x(0) = 0, \quad x(T) = B, \quad u(t) \ge 0. \qquad (12)$$

The initial state (inventory level) is zero and is supposed to reach B by time T. Inventory holding cost accrues at c_2 per unit and production cost increases with the square of the production rate. We form the Hamiltonian:

$$H = c_1 u^2 + c_2 x + \lambda u.$$

Then

$$\partial H / \partial u = 2 c_1 u + \lambda = 0, \qquad (13)$$

$$\lambda' = -\partial H / \partial x = -c_2. \qquad (14)$$

To find x, u, λ that satisfy (12)–(14), integrate (14) to get λ and substitute into (13). Then put u into (12) and integrate. The two constants of integration are found using the boundary conditions, yielding

$$x(t) = c_2 t(t - T)/4c_1 + Bt/T,$$

$$u(t) = c_2(2t - T)/4c_1 + B/T,$$

$$\lambda(t) = c_2 T/2 - 2c_1 B/T - c_2 t,$$

so long as $u \ge 0$ for $0 \le t \le T$. This is the solution obtained by the calculus of variations, as it should be.

ɜatisfy

, $m' = (r - 1)m.$

the system in a. is (0, 0) and it is totally

$= x_0 e^t, \qquad m(t) = 0, \qquad J = 0.$

Gould. Exercise 2 is discussed in detail by
Colin Clark, whereas Exercise 5 is discussed

imal solution in the neighborhood of a steady
is more than one state variable. The class of

$(x, u)\, dt$

$= u, \quad x(0) = x_0,$

$u_n]$ and F is a twice differentiable concave
ɪdied by means of a linear approximation in
ıari and Liviatan have shown that if F is
if m_i is a root of the characteristic equation,
ability is impossible; if the real part of m_i is
nnot be negative. Further, they have shown
ıen there are no purely imaginary roots. See

f an equilibrium. Under what conditions will
icular equilibrium regardless of the initial
Ieal for an extensive analysis of a particular
ɔck (1977) for a survey of results on global
Magill (1977a, b), Cass and Shell, Brock and
n (1978) for representative papers on stabil-

ɔncept and of the existence and properties of
e been addressed by Halkin (1974), Haurie
howed that in general there are no necessary
horizon problem. See also Seierstad and

Section 10

Bounded Controls

The control may be bounded, as in

$$\max \int_{t_0}^{t_1} f(t, x, u)\, dt \tag{1}$$

$$\text{subject to} \qquad x' = g(t, x, u), \quad x(t_0) = x_0, \tag{2}$$

$$a \le u \le b. \tag{3}$$

Absence of a bound is a special case with either $a \to -\infty$ or $b \to \infty$, as appropriate. For instance, gross investment may be required to be nonnegative.

Let J denote the value of the integral in (1). After appending (2) with a multiplier and integrating by parts, one can compute the variation δJ, the linear part of $J - J^*$.

$$\delta J = \int_{t_0}^{t_1} \left[(f_x + \lambda g_x + \lambda')\, \delta x + (f_u + \lambda g_u)\, \delta u \right] dt - \lambda(t_1)\, \delta x(t_1). \tag{4}$$

Choose λ to satisfy

$$\lambda' = -(f_x + \lambda g_x), \qquad \lambda(t_1) = 0, \tag{5}$$

so that (4) reduces to

$$\delta J = \int_{t_0}^{t_1} (f_u + \lambda g_u)\, \delta u\, dt. \tag{4'}$$

In order for x, u, λ to provide an optimal solution, no comparison path can yield a larger value to the objective. Thus,

$$\delta J = \int_{t_0}^{t_1} (f_u + \lambda g_u)\, \delta u\, dt \le 0 \tag{6}$$

is required for all feasible modifications δu. Feasible modifications are those that maintain (3). If the optimal control is at its lower bound a at some t, then the modified control $a + \delta u$ can be no less than a for feasibility, so $\delta u \geq 0$ is required. Similarly, if the optimal control is at its upper bound b, then any feasible modification satisfies $\delta u \leq 0$. Summarizing,

$$\begin{aligned} &\delta u \geq 0 \quad \text{whenever} \quad u = a, \\ &\delta u \leq 0 \quad \text{whenever} \quad u = b, \\ &\delta u = \text{unrestricted} \quad \text{whenever} \quad a < u < b. \end{aligned} \tag{7}$$

We need (6) to be satisfied for all δu consistent with (7). Therefore, u will be chosen so that

$$\begin{aligned} u(t) &= a \quad &\text{only if} \quad f_u + \lambda g_u \leq 0 \quad &\text{at} \quad t, \\ a < u(t) &< b \quad &\text{only if} \quad f_u + \lambda g_u = 0 \quad &\text{at} \quad t, \\ u(t) &= b \quad &\text{only if} \quad f_u + \lambda g_u \geq 0 \quad &\text{at} \quad t. \end{aligned} \tag{8}$$

For instance, if $u^*(t) = a$, then (from (7)) $\delta u \geq 0$ is required, and thus $(f_u + \lambda g_u)\,\delta u \leq 0$ only if $f_u + \lambda g_u \leq 0$. Similarly, if $u^*(t) = b$, then $\delta u \leq 0$ is required for a feasible modification and thus $(f_u + \lambda g_u)\,\delta u \leq 0$ only if $f_u + \lambda g_u \geq 0$. And, as usual, if $a < u^*(t) < b$, then δu may have any sign so that $(f_u + \lambda g_u)\,\delta u \leq 0$ can be assured only if $f_u + \lambda g_u = 0$ at t. A statement equivalent to (8) is

$$\begin{aligned} f_u + \lambda g_u < 0 \quad &\text{implies} \quad u(t) = a, \\ f_u + \lambda g_u = 0 \quad &\text{implies} \quad a \leq u(t) \leq b, \\ f_u + \lambda g_u > 0 \quad &\text{implies} \quad u(t) = b. \end{aligned} \tag{8$'$}$$

Thus, if x^*, u^* solve (1)–(3), then there must be a function λ such that x^*, u^*, λ satisfy (2), (3), (5), and (8). These necessary conditions can be generated by means of the Hamiltonian

$$H = f(t, x, u) + \lambda g(t, x, u).$$

Then (2) and (5) result from

$$x' = \partial H/\partial \lambda, \qquad \lambda' = -\partial H/\partial x.$$

Conditions (8) can be generated by maximizing H subject to (3); this is an ordinary nonlinear programming problem in u.

Solve

$$\begin{aligned} &\max \quad H = f + \lambda g \\ &\text{subject to} \quad a \leq u \leq b \end{aligned} \tag{9}$$

by appending the constraints to the objective with multipliers w_1, w_2. The Lagrangian for (9) is (see Section A6)

$$L = f(t, x, u) + \lambda g(t, x, u) + w_1(b - u) + w_2(u - a), \tag{10}$$

from which we obtain the necessary
with respect to u:

$$\partial L/\partial u = f_u + \lambda g$$

$$w_1 \geq 0,$$

$$w_2 \geq 0,$$

Conditions (11)–(13) are equivaler
alternative statement of the requireme
$u^*(t) = a$, then $b - u^* > 0$, so (12
$f_u + \lambda g_u + w_2 = 0$. Since $w_2 \geq 0$,
This is the first instance in (8). One
possibilities.)

Example 1. We solved our production

$$\min \int_0^T (c_1 u^2 + c_2 x)\, dt$$

subject to $x' = u, \quad x(0)$

in Section 6 and elsewhere in the case
that plan is not feasible and explicit acc
constraint $u \geq 0$. We now discuss this
This control $u(t)$ is to be chosen at e

$$H = c_1 u^2 + c_2 x + \lambda u,$$

The Lagrangian, with multiplier functio

$$L = c_1 u^2 + c_2$$

Necessary conditions for u to be minim

$$\partial L/\partial u = 2 c_1 u$$

$$w \geq 0, \qquad u \geq$$

Further,

$$\lambda' = -\partial H/\partial$$

so that

$$\lambda(t) = k$$

for some constant k_0. Substituting from

$$u(t) = (w - \lambda)/2c_1 =$$

To solve, we make a conjecture about
seek a path with this structure satisfying

s δu. Feasible modifications are those
is at its lower bound a at some t, then
less than a for feasibility, so $\delta u \geq 0$ is
trol is at its upper bound b, then any
Summarizing,

$u = a,$
$u = b,$
whenever $\quad a < u < b. \qquad (7)$

u consistent with (7). Therefore, u will

f $\quad f_u + \lambda g_u \leq 0 \quad$ at $\quad t,$
f $\quad f_u + \lambda g_u = 0 \quad$ at $\quad t,$
f $\quad f_u + \lambda g_u \geq 0 \quad$ at $\quad t. \qquad (8)$

om (7)) $\delta u \geq 0$ is required, and thus
$_u \leq 0$. Similarly, if $u^*(t) = b$, then
dification and thus $(f_u + \lambda g_u)\,\delta u \leq 0$
, if $a < u^*(t) < b$, then δu may have
n be assured only if $f_u + \lambda g_u = 0$ at t.

iplies $\quad u(t) = a,$
iplies $\quad a \leq u(t) \leq b,$
iplies $\quad u(t) = b. \qquad (8')$

n there must be a function λ such that
8). These necessary conditions can be
n

$) + \lambda g(t, x, u).$

$\lambda' = -\partial H/\partial x.$

naximizing H subject to (3); this is an
lem in u.

$= f + \lambda g$
$a \leq u \leq b \qquad (9)$

objective with multipliers w_1, w_2. The
)

$u) + w_1(b - u) + w_2(u - a), \qquad (10)$

from which we obtain the necessary conditions for a constrained maximum with respect to u:

$$\partial L/\partial u = f_u + \lambda g_u - w_1 + w_2 = 0, \tag{11}$$

$$w_1 \geq 0, \qquad w_1(b - u) = 0, \tag{12}$$

$$w_2 \geq 0, \qquad w_2(u - a) = 0. \tag{13}$$

Conditions (11)–(13) are equivalent to conditions (8) and constitute an alternative statement of the requirement, as will be shown in Exercise 6. (If $u^*(t) = a$, then $b - u^* > 0$, so (12) requires $w_1 = 0$; hence, from (11), $f_u + \lambda g_u + w_2 = 0$. Since $w_2 \geq 0$, we have $f_u + \lambda g_u \leq 0$ if $u^*(t) = a$. This is the first instance in (8). One continues similarly for the other two possibilities.)

Example 1. We solved our production planning problem

$$\min \int_0^T \left(c_1 u^2 + c_2 x\right) dt$$

$$\text{subject to} \qquad x' = u, \quad x(0) = 0, \quad x(T) = B, \quad u(t) \geq 0$$

in Section 6 and elsewhere in the case of $B \geq c_2 T^2/4c_1$. If $B < c_2 T^2/4c_1$, that plan is not feasible and explicit account must be taken of the nonnegativity constraint $u \geq 0$. We now discuss this case.

This control $u(t)$ is to be chosen at each t to minimize the Hamiltonian

$$H = c_1 u^2 + c_2 x + \lambda u, \qquad \text{subject to} \quad u \geq 0.$$

The Lagrangian, with multiplier function w, is

$$L = c_1 u^2 + c_2 x + \lambda u - wu.$$

Necessary conditions for u to be minimizing (see Exercise 1) are

$$\partial L/\partial u = 2c_1 u + \lambda - w = 0, \tag{14}$$

$$w \geq 0, \qquad u \geq 0, \qquad wu = 0. \tag{15}$$

Further,

$$\lambda' = -\partial H/\partial x = -c_2,$$

so that

$$\lambda(t) = k_0 - c_2 t \tag{16}$$

for some constant k_0. Substituting from (16) into (14) and rearranging gives

$$u(t) = (w - \lambda)/2c_1 = (c_2 t - k_0 + w)/2c_1. \tag{17}$$

To solve, we make a conjecture about the structure of the solution and then seek a path with this structure satisfying the conditions. Since the time span T

is long relative to the amount B to be produced, we guess that there is an initial period, say $0 \le t \le t^*$ (for some t^* to be determined), with no production or inventory. Production begins at t^*. Thus our hypothesis is

$$\begin{aligned} u(t) &= 0, \qquad 0 \le t < t^*, \\ u(t) &> 0, \qquad t^* \le t \le T, \end{aligned} \tag{18}$$

for some t^* to be determined.

When $u(t) = 0$, we have from (17)

$$w(t) = k_0 - c_2 t \ge 0, \qquad 0 \le t < t^*. \tag{19}$$

Nonnegativity in (19) is required by (15). From (19), $w(t)$ decreases on $0 \le t < t^*$, so nonnegativity is assured provided

$$k_0 - c_2 t^* \ge 0. \tag{20}$$

When $u(t) > 0$, (15) implies $w(t) = 0$. Then from (17)

$$u(t) = (c_2 t - k_0)/2c_1 \ge 0, \qquad t^* \le t \le T. \tag{21}$$

Since $u(t)$ increases after t^*, $u(t) \ge 0$ for $t^* \le t \le T$ provided that $u(t^*) = (c_2 t^* - k_0)/2c_1 \ge 0$. This requirement and (20) together imply that

$$k_0 = c_2 t^*. \tag{22}$$

Hypothesis (18) now takes the more concrete form

$$\begin{aligned} u(t) &= 0, \qquad 0 \le t < t^*, \\ u(t) &= c_2(t - t^*)/2c_1, \qquad t^* \le t \le T. \end{aligned} \tag{23}$$

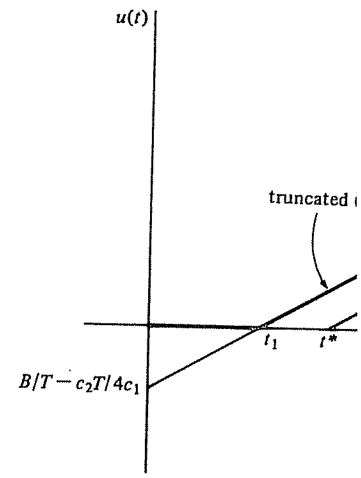
Recalling that $u = x'$ and integrating yields

$$\begin{aligned} x(t) &= 0, \qquad 0 \le t < t^*, \\ x(t) &= c_2(t - t^*)^2/4c_1, \qquad t^* \le t \le T. \end{aligned} \tag{24}$$

The constants of integration were evaluated using the final condition $x(0) = 0$ and the required continuity of x (so $x(t^*) = 0$). Finally, combining (24) with the terminal condition $x(T) = B$ gives

$$t^* = T - 2(c_1 B/c_2)^{1/2}. \tag{25}$$

With a distant delivery date T, the duration of the production period $T - t^*$ varies directly with $c_1 B/c_2$; it increases with the amount to be produced and the production cost coefficient c_1 and decreases with the unit holding cost c_2. It is precisely the period obtained under the supposition that production had to begin immediately but that the delivery date T could be chosen optimally; see Example I9.1.



Figur

In sum, the solution is given in the a
by (25). Extending our sufficiency th
region shows the solution tabulated to b

	$0 \le t < t^*$
$u(t)$	0
$x(t)$	0
$\lambda(t)$	$c_2(t^* - t)$
$w(t)$	$c_2(t^* - t)$

Observe that the solution could *not* be
unconstrained problem

$$u(t) = c_2(2t - $$

and deleting the nonfeasible portion! Ta
$B \ge c_2 T^2/4c_1$, and setting $u = 0$ whene
the optimal solution. This is algebraicall
cally in Figure 10.1 where $t_1 = T$
$4c_1 B/c_2 t$, and t^* is given in (25).

In Figure 10.1, the "solution" (26) be
t_1. Since output is negative for $0 \le t <$
Production from t_1 until t_2 is devoted t
Production from t_2 until T fulfills th
production were to begin at t_2 and foll
would be produced but costs would be n
plan is reflected in the optimal path.

be produced, we guess that there is an
some t^* to be determined), with no
egins at t^*. Thus our hypothesis is

$0 \le t < t^*,$

$t^* \le t \le T,$ (18)

$\ge 0, \quad 0 \le t < t^*.$ (19)

' (15). From (19), $w(t)$ decreases on
d provided

${}_2 t^* \ge 0.$ (20)

0. Then from (17)

$_1 \ge 0, \quad t^* \le t \le T.$ (21)

) for $t^* \le t \le T$ provided that $u(t^*) =$
nt and (20) together imply that

$c_2 t^*.$ (22)

oncrete form

$< t^*,$

$2c_1, \quad t^* \le t \le T.$ (23)

rields

$< t^*,$

$4c_1, \quad t^* \le t \le T.$ (24)

iated using the final condition $x(0) = 0$
$(t^*) = 0)$. Finally, combining (24) with
s

$(c_1 B/c_2)^{1/2}.$ (25)

ration of the production period $T - t^*$
es with the amount to be produced and
decreases with the unit holding cost c_2.
r the supposition that production had to
' date T could be chosen optimally; see

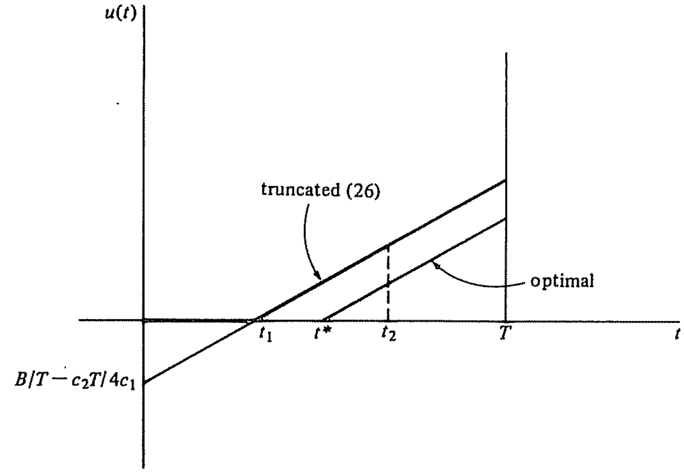


Figure 10.1

In sum, the solution is given in the accompanying table, where t^* is given by (25). Extending our sufficiency theorem to cover a constrained control region shows the solution tabulated to be optimal (see Section 15).

	$0 \le t < t^*$	$t^* \le t \le T$
$u(t)$	0	$c_2(t - t^*)/2c_1$
$x(t)$	0	$c_2(t - t^*)^2/4c_1$
$\lambda(t)$	$c_2(t^* - t)$	$c_2(t^* - t)$
$w(t)$	$c_2(t^* - t)$	0

Observe that the solution could *not* be obtained by taking the solution to the unconstrained problem

$$u(t) = c_2(2t - T)/4c_1 + B/T \tag{26}$$

and deleting the nonfeasible portion! Taking this solution, appropriate only if $B \ge c_2T^2/4c_1$, and setting $u = 0$ whenever it dictates $u < 0$ does not provide the optimal solution. This is algebraically clear and is also illustrated graphically in Figure 10.1 where $t_1 = T/2 - 2c_1B/c_2T$, $t_2 = 2t_1 = T - 4c_1B/c_2t$, and t^* is given in (25).

In Figure 10.1, the "solution" (26) begins with $u < 0$ and reaches $u = 0$ at t_1. Since output is negative for $0 \le t < t_1$, inventory is likewise negative. Production from t_1 until t_2 is devoted to driving inventory back up to zero! Production from t_2 until T fulfills the total delivery requirement B. If production were to begin at t_2 and follow the truncated path, a total of B would be produced but costs would be needlessly high. The cost minimizing plan is reflected in the optimal path.