Homework #7
APM111: Introduction to Numerical Methods
Due: April 18, 2006

1. For the test problem $\frac{dy}{dt} = \lambda y$, where λ is a complex number, calculate and plot the stability region of the trapezoidal method

$$y_{n+1} = y_n + \frac{\Delta t}{2}\left[f(t_n, y_n) + f(t_{n+1}, y_{n+1})\right]$$

and the Heun's method

$$y_{n+1} = y_n + \frac{\Delta t}{2}\left[f(t_n, y_n) + f\left(t_{n+1}, y_n + f(t_n, y_n)\Delta t\right)\right]$$

The stability region is the region in the complex plane of $z = \lambda \Delta t$ where y_n does not go to infinity as t goes to infinity. Identify which method is an implicit method and which method is an explicit method. Does the implicit method have a bigger or smaller stability region compared to its explicit counterpart?

2. Show algebraically that the Heun's method is 2^{nd} order accurate, i.e. the local discretization error is $O(\Delta t^3)$, where Δt is the time step. Note that since f depends on both t and y, you need to use Taylor expansion in two-dimensions. The trapezoidal method can be shown similarly to be 2^{nd} order accurate as well, but we will not bore you with a second derivation.

3. Implement the trapezoidal method and the Heun's method and use them to solve the test problem (see problem 1) with $\lambda = 2i$, $i = \sqrt{-1}$ and $y(t=t_0)=1$. Repeat your experiments with different time steps to check your results from problems 1 and 2 experimentally. To check your results from problem 1, try $\Delta t =$ 0.5, 0.2, and 0.1, and integrate over the time interval [0, 50]. To show the methods are 2^{nd} order accurate, use $\Delta t = 0.1/2^k, k = 0,1,2,...9$ and integrate over the time interval [0, 3]. Find your global errors by comparing your numerical results with the exact solution at the end of the simulation (i.e. t=3). Plot the results as log-log plots. With the small time steps, the runs may take a few minutes.

4. Solve the single pendulum problem

$$\ddot{\theta} = -\sin\theta$$
$$\theta(t=0) = \theta_0$$
$$\dot{\theta}(t=0) = 0$$

for the time interval [0, 20π] with your favorite Matlab ODE solver for non-stiff problems. Do it for θ_0=0.1, π-0.1, and π, and plot your results. For the last case, does the numerical solution match the exact solution? And why?