

Final Projects

Applied math 111: numerical methods

Due: end of reading period, May 17, 2006

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1 Introduction

Folks, here is the list of final projects. Please read it carefully, and come to talk to one of us (whose name appears on the project) about your choice before you start. We'd like to also meet with you mid-way along your project, during the week of May 8, to see if all goes well and if there is anything we can do to help. And as we mentioned in class, if you'd like to come up with your own project, you are welcome to do so, please let me know and come talk to me about the details.

In all projects there are some unspecified details such as values of some constants, number of points in a finite difference grid, missing physical assumptions, etc. You may assume whatever you find convenient in order to complete your project, and feel free to consult us on such issues.

You need to submit a 2 page report summarizing your findings, including all the relevant plots, as well as the Matlab code used for the project. **Please email your report to Mike.**

Please remember, this isn't meant to be an exam for you to do on your own, but rather a project to do with our close help and supervision (if needed). We want to make sure you all enjoy and succeed in these projects and are available to help.

Enjoy, Eli

2 Predicting the stock market

(Eli; Markov process, least squares, SVD)



The purpose here is to try and predict the stock market and to test how good the prediction is compared to some simple assumptions that one could make on the stock market behavior. Successfully predicting the stock prices is, unfortunately, not a trivial problem as you could guess. Still, approaches very roughly similar (although much more sophisticated) to the one we take here are used by various financial institutions.

1. Start by reading the data, which is given in the file `stock_prices.dat` in the supporting materials directory under sub directory `exam`, [here](#). We will only use the data from 1997-2006 in this project.
2. Assume that we can predict the stock prices today from their prices during the past 10 days. Based on this assumption, fit a 10th order Markov process

$$\theta_n = \sum_{i=1}^{10} \beta_i \theta_{n-i}$$

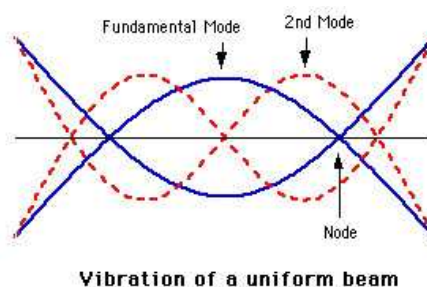
to the data between 1997 and 2003, finding the coefficients β_i by solving the appropriate least squares. Solve the problem by using SVD and generalized (pseudo) inverse (That is, use Matlab's SVD routine to calculate U, Σ, V , construct the pseudo inverse and use it to solve the equations. Compare the results to that obtained using Matlab's backslash operator). Plot the singular values σ_i of the corresponding matrix in decreasing order as function of i . Beware of zero singular values if any when you construct your pseudo inverse. You can compare to what Matlab's `pinv` does as a debugging aid.

3. The Markov process relates the stock values today to those in the past 10 days. Using this Markov process, go over all days from 2004-2006 and predict the stock prices for each day during the period based on the previous 10 days.

4. Let the prediction for a given day be θ_i and the actual stock prices be θ_i^* . Calculate the residuals (error) from the prediction, $r_i = \theta_i - \theta_i^*$ and calculate both the standard deviation of the error and its auto correlation $\langle r_i r_{i+n} \rangle$, where $\langle \rangle$ denotes average over i . Evaluate this autocorrelation for $n = 0, 1, 2, 3, \dots, 20$. Is the error a white noise? (what do you expect the auto correlation of white noise to be?)
5. Try predicting the stocks using random numbers that have the same mean and standard deviation as the observed prices. Calculate and characterize/ plot the residual as in the previous section.
6. Try predicting the prices using “persistence”. that is, assume that the prices at each day are just equal to those at the previous day. Calculate and characterize/ plot the residual as in the previous section.
7. Summarize your results. Which prediction approach is best? Do you feel prepared to go into financial advising now?

3 Good vibrations

(Eli; eigenvalues/ vectors/ wave equation and animation!)



Consider the latitudinal vibrations of a string with non-uniform density. The purpose of the project is to calculate the possible frequencies of the string, and then also solve the governing partial differential wave equation and animate(!) the string motion as function of time. See waves propagating along the string, and earn the eternal glory of having your animation posted on the course homepage.

The equation for the latitudinal vibrations of a string of is the same wave equation derived in class for the longitudinal waves:

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x} \left(c^2 \frac{\partial u}{\partial x} \right)$$

where $c(x)$ is the velocity of waves along the string. Because we assume the string's density is not uniform, the velocity is a function of distance along the string, x . The boundary conditions are that the string cannot move at the edges,

$$u(x = 0, t) = u(x = L, t) = 0$$

1. **Vibration modes and characteristic frequencies:** Show that assuming a solution of the form

$$u(x, t) = u_k(x) \cos(\omega_k t)$$

and using a finite difference approximation in x , the equation is (similar to that derived in class)

$$-\omega_k^2 u_k(x_i) = = \frac{1}{(\Delta x)^2} \left[\left\{ \frac{1}{2} (c^2(x_{i+1}) + c^2(x_i)) (u_k(x_{i+1}) - u_k(x_i)) \right\} - \left\{ \frac{1}{2} (c^2(x_i) + c^2(x_{i-1})) (u_k(x_i) - u_k(x_{i-1})) \right\} \right],$$

where $x_i = i\Delta x$ with $i = 0, \dots, N$ and $(N - 1)\Delta x = L$.

2. Solve the above eigenvalue/ eigenvector problem for the following cases

(a) $c^2(x) = c_0^2 = \text{constant}$.

(b) A Gaussian-shaped velocity profile

$$c^2(x) = c_0^2 \left[1 + \exp \left(-\frac{(x-L/2)^2}{(L/5)^2} \right) \right].$$

(c) A velocity with a “break” profile

$$c^2(x) = c_0^2 \left[1 + 0.5 \tanh \left(-\frac{(x-L/2)}{(L/5)} \right) \right].$$

for each case, plot the first 10 eigenvalues (frequencies) and eigenvectors. discuss your results and the difference between the different $c^2(x)$ profiles. Make sure you use a large enough number of grid points N . To check, increase N by a factor of two and see that the results don't change significantly.

3. Calculate the expected sensitivity of the eigenvalues of this problem using the left eigenvectors approach. Do the calculation explicitly rather than using the built in Matlab function `condeig`. Plot the sensitivity of the eigenvalues as error bars on the eigenvalue plots requested above, assuming a perturbation size of 1% for the elements of the matrix.
4. **Explicit time dependent solution:** This is a fun little problem. My program which solves this section is only some 30 lines long in Matlab, and the animation is quite fun. The wave equation with a constant wave velocity may be approximated using finite difference in space and time as

$$\begin{aligned} & \frac{1}{(\Delta t)^2} \left[u(x_i, t_{n+1}) - 2u(x_i, t_n) + u(x_i, t_{n-1}) \right] \\ &= \frac{1}{(\Delta x)^2} \left[c^2 \{ u(x_{i+1}, t_n) - 2u(x_i, t_n) + u(x_{i-1}, t_n) \} \right] \end{aligned}$$

this means that given the solution at times n and $n - 1$, we can calculate it at all grid point at time $n + 1$ using the above equation. Use this to integrate the equation in time. Note that you only need to calculate the solution at the interior points, as the solution is zero at the ends for all times because of the boundary conditions.

5. Start with initial conditions corresponding to the pulling of the string in its mid-point, and calculate the evolution for some time after that. Plot the solution every several time steps, using `pause(0.1)` after each plot, to effectively create an animation.
6. **Optional:** save the animation of the resulting string motion in one of the movie formats available in Matlab which can be played outside of Matlab using some standard viewer. Submit your movie entry to be displayed on the course home page.

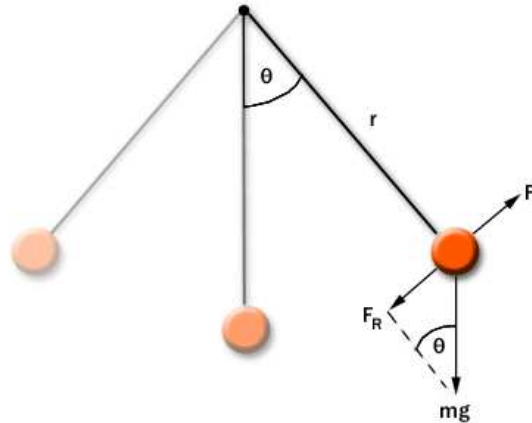
Some technical comments: (1) it is advisable to keep in your program three arrays, for the string displacement $u(x_i)$ at the present time, previous time, and next time step ($n - 1, n, n + 1$). Assuming these arrays are named `u_prev`, `u_now`, `u_next`, at the end of each time step, after calculating `u_next(i)`, set for all spatial locations (index i):

`u_prev(i)=u_now(i); u_now(i)=u_next(i)`

in preparation for the next time step. **(2)** at the first time step, $n = 1$, you need to specify two previous time steps to be able to solve for the next time. Please assume that the solution at $n = 0$ and $n = -1$ are both equal to the initial conditions you specify. **(3)** Use 200 grid points in space, $L = 1$, $dt = 0.001$, $c^2 = 2$.

4 A driven pendulum

(Eli, ODEs)



This is a problem for those of you who are interested in nonlinear dynamics. A simple pendulum can show very surprising behaviors when it is not assumed to be making small oscillations and nonlinearity is included. The reference is Strogatz, *nonlinear dynamics and chaos*, section 8.5 pages 265-272; you don't need to understand this entire section in Strogatz to do this project, but it may still be useful to read it. Consider a pendulum composed of a stiff weightless rod with a mass m at its end, driven by a constant torque that is applied to the rod. The governing equation is

$$\frac{d^2\theta}{dt^2} + \alpha \frac{d\theta}{dt} + \sin\theta = I$$

where θ is the angle between the vertical and the pendulum, and I is the constant torque. The term with α represents the friction felt by the pendulum. The $\sin\theta$ term represents the force of gravity.

The purpose of the project is to identify all possible behaviors of the pendulum that are encountered for different values of the driving torque I and the friction coefficient α .

1. Justify the above equation from a physical point of view, explaining (heuristically but explicitly) how it is related to Newton's law of motion $F = ma$, and what is the physical justification for each term in the equation.
2. Rewrite the equation as a set of two first order ODEs for θ and $\dot{\theta} = d\theta/dt$
3. **Preliminary solution:** write a code that integrates these ODEs using a Heun method which you implement yourself in Matlab. Heun's method is the one that is analogous to the trapezoidal rule, given on page 192 in Moler's book (page 6 in the on-line version of this chapter). Solve with $\alpha = 1.0$ and with both $I = 0.5$ and $I = 1.5$. Use as initial conditions $\theta = \pi/4$ and $\dot{\theta} = 0$. Plot the time series for θ and $\dot{\theta}$ and plot the solution as a trajectory in a two dimensional space whose coordinates are θ (remembering that physically theta is in the range $[0, 2\pi]$ and plotting only in that range by plotting $\text{mod}(\theta, 2\pi)$) and $\dot{\theta}$. Explain your results physically.

4. Steady states:

- (a) Are there any steady states solutions in which the pendulum does not move? Find any such states and interpret the physical balances of forces.
- (b) Find if the steady states are stable: start your Matlab solver with initial conditions very near these steady states and integrate the equations numerically to find out if the solution gets back to these steady states (in which case they are “stable”) or deviates from them (“unstable”). Plot the time series for θ and $\dot{\theta}$ and plot the solution as a trajectory in a two dimensional space whose coordinates are θ and $\dot{\theta}$. Explain your results physically.

5. Exploring the parameter regime:

- (a) Solve the equations for different values of $0 < \alpha < 2$ and $0 < I < 2$ (use, say, a 10×10 grid or so in the (α, I) plane). Always start your pendulum from rest at $\theta = 0$. Plot a schematic diagram in the (α, I) plane explaining which solutions exist at each point. Try to draw the boundaries of these regions.
- (b) At each of the points in the above grid, check if the steady states you found in the above section exist and are stable.

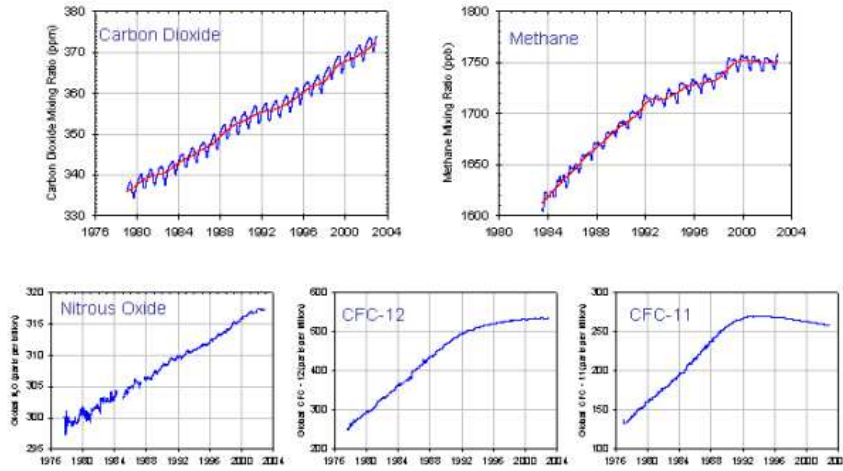
6. Hysteresis:

- (a) Set $\alpha = 0.3$, $I = 0$. Integrate the equations until a steady state is approached (what is it? why?).
- (b) While the equations are integrated, *very slowly* increase I from zero to 2 and back to zero such that it is increasing and then decreasing linearly in time. Do you observe any change in the behavior for a given value of I while it is increased as compared to while it is decreased? This phenomena is referred to as hysteresis.
- (c) Plot a time series of your model results in a way that demonstrates the hysteresis.
- (d) How would you change your schematic plot of the behavior in different parameter regimes given this hysteresis?

5 Kyoto protocol

(Mike; Shooting Problem)

Global Trends in Major Greenhouse Gases to 1/2003



Global trends in major long-lived greenhouse gases through the year 2002. These five gases account for about 97% of the direct climate forcing by long-lived greenhouse gas increases since 1750. The remaining 3% is contributed by an assortment of 10 minor halogen gases, mainly HCFC-22, CFC-113 and CCl_4 .

Suppose that a new Kyoto-style treaty is being prepared that not only promises to lower emissions of carbon into the atmosphere, but also to achieve a specific partial pressure of atmospheric CO_2 by the year 2100. Your job is to ensure that the treaty's goals are possible, using a model that all the participating countries have agreed upon¹.

The model being used is as follows. Defining the following eight variables:

p , partial pressure of carbon dioxide in the atmosphere

σ_s , total dissolved carbon concentration in the shallow ocean

σ_d , total dissolved carbon concentration in the deep ocean

α_s alkalinity in the shallow ocean

α_d alkalinity in the deep ocean

h_s , hydrogen carbonate in the shallow ocean

c_s , carbonate in the shallow ocean

p_s , partial pressure of gaseous carbon dioxide in the shallow ocean.

We must also include the effects of burning fossil fuels, and define $f(t)$ as the rate at which carbon is sent into the atmosphere from human activities.

The system is governed by the following differential equations

$$\frac{dp}{dt} = \frac{p_s - p}{d} + \frac{f(t)}{\mu_1} \quad (1)$$

$$\frac{d\sigma_s}{dt} = \frac{1}{v_s} \left((\sigma_d - \sigma_s)w - k_1 - \frac{p_s - p}{d} \mu_2 \right) \quad (2)$$

$$\frac{d\sigma_d}{dt} = \frac{1}{v_d} (k_1 - (\sigma_d - \sigma_s)w) \quad (3)$$

¹Of course, the model may or may not be exact, but if your predictions are wrong because of the model nobody will blame you (they promise!)

$$\frac{d\alpha_s}{dt} = \frac{1}{v_s}((\alpha_d - \alpha_s)w - k_2) \quad (4)$$

$$\frac{d\alpha_d}{dt} = \frac{1}{v_d}(k_2 - (\alpha_d - \alpha_s)w), \quad (5)$$

along with three nonlinear conditions:

$$h_s = \frac{\sigma_s - \sqrt{\sigma_s^2 - k_3\alpha_s(2\sigma_s - \alpha_s)}}{k_3} \quad (6)$$

$$c_s = \frac{\alpha_s - h_s}{2} \quad (7)$$

$$p_s = k_4 \frac{h_s^2}{c_s}. \quad (8)$$

Strictly speaking, these variables also follow their own dynamic rate equations, but relax to their steady state values far faster than the other five variables. Because the equilibrium points of the dissolved gases are determined by the more slowly-varying variables, and we are interested in long time-scales, we can ignore their fast dynamics.

The best efforts of modern science^{2 3} have determined the relevant constants to be

$$\begin{aligned} d &= 8.64 \\ \mu_1 &= 4.95 \times 10^2 \\ \mu_2 &= 4.95 \times 10^{-2} \\ v_s &= 0.12 \\ v_d &= 1.23 \\ w &= 10^{-3} \\ k_1 &= 2.19 \times 10^{-4} \\ k_2 &= 6.12 \times 10^{-5} \\ k_3 &= 0.997148 \\ k_4 &= 6.79 \times 10^{-2}. \end{aligned}$$

In the year 999, Sigmundur Brestisson began measuring carbon concentrations. Because he was busy Christianizing the Faroe Islands, he didn't get a complete set of measurements until the year 1000. These were:

$$\begin{aligned} p &= 1.00 \\ \sigma_s &= 2.01 \\ \sigma_d &= 2.23 \\ \alpha_s &= 2.20 \\ \alpha_d &= 2.26. \end{aligned}$$

Unfortunately, Sigmundur was tragically killed shortly after swimming from irate Christians to the island of Suouroy, and no further measurements were taken.

²Yes, we've stolen the parameters from Moler's textbook –but he got them from somewhere else³ too!

³J. C. G. Walker, Numerical adventures with geochemical cycles, Oxford University Press, New York, 1991.

Careful measurements of the total man-made emission rate of carbon into the atmosphere, begun by monks in the Alps, and finished by Nostradamus, indicate that $f(t)$ can be interpolated⁴ from

year	rate
1000	0.0
1850	0.0
1950	1.0
1980	4.0
2000	5.0
2050	8.0.

After much debate and media coverage, there is an agreement to cut CO₂ emissions by 95% from Nostradamus' prediction for 2050, and that the partial pressure of CO₂ in the atmosphere should be back to the level it was at in 1906. Because the monks worked so slowly at measuring humanity's CO₂ output, it is your job to interpolate from their data the function $f(t)$, and use this to calculate the partial pressure of CO₂ in the atmosphere in 1906. You must then determine whether cutting emissions by 95% by 2050 (more interpolation!) will allow the world to achieve this goal, or if it's time to go back to the drawing board.

Note that you will have to determine a reasonable form for the function $f(t)$ from now until 2050. To find this reasonable form, assume that all people in the world produce roughly the same contribution to $f(t)$, and that the world population follows the Logistic Equation

$$\frac{dN}{dt} = \frac{rN(K - N)}{K}. \quad (9)$$

Choose a reasonable value for the maximum population growth rate r and the carrying capacity K . Determine what kinds of sacrifices are likely to be necessary on each person's part to achieve our CO₂ goals. Is it necessary to adjust r ? Describe some suitably iron-fisted regime to enforce upon the hapless people of the world so as to make the carbon output rate $f(t)$ have the properties you need to meet the treaty's conditions. Be as creative as possible.

Drawing Board

On the drawing board, we find plans for a vast Ocean Mixing Generator (OMG), designed to increase the effective mixing rate w of carbon in the deep and shallow oceans. In conjunction with LALA, the Large Atmosphere-Liquid Alternator, which decreases d , it is hoped that the world will finally be able to achieve the atmosphere of 1906. Unfortunately, operation of these megascale devices will only allow emissions to be cut by 85% of Nostradamus' 2050 predictions if one is used, and by 80% if both are used. (The operation of these devices produces CO₂ independently of that produced by other human activity)

Determine whether tuning w or d separately by use of OMG or LALA will make it possible to achieve the treaty's goals. Determine all values (if any) of d and w which obey the treaty's demands, and plot them on a map of Guam.

⁴Yes, you do need to interpolate it!