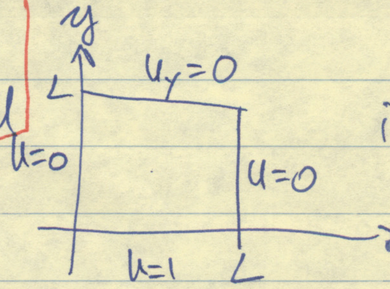
1

$u_t + \hat{U} u_x = \alpha^2 (u_{xx} + u_{yy}) - \gamma \cdot u$

advection, diffusion & cooling of a square plate.



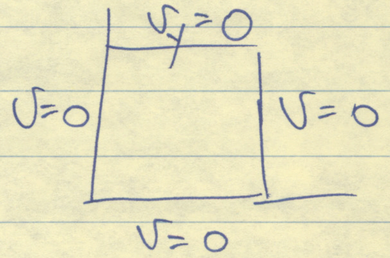
i.c.: $u(x,y,t=0) = f(x,y)$

solve first steady problem: $u = \bar{u}(x,y)$

then write $u = \bar{u} + v(x,y,t)$.

v satisfies same eq'n as u, but homog. b.c.:

$\bar{u}$ satisfies same b.c. as u.



So: $\bar{u} = X \cdot Y$

$$\Rightarrow \hat{U}\frac{X'}{X} - \alpha^2 \frac{X''}{X} = k^2 - \gamma \Rightarrow \boxed{X'' - U X' + \frac{\hat{k}^2}{\alpha^2} X = 0}$$

let $\hat{k}^2 \equiv k^2 - \gamma$ $\qquad$ $U \equiv \hat{U}/\alpha^2$

$$\alpha^2 \frac{Y''}{Y} = -k^2 \Rightarrow \boxed{Y'' = +\frac{k^2}{\alpha^2} Y}$$

$$Y(y) = A\cosh\left(\frac{k}{\alpha} y\right) + B\sinh\left(\frac{k}{\alpha} y\right) + Cy + D$$

$$X = e^{\lambda x} \Rightarrow \lambda^2 - \lambda U + \frac{\hat{k}^2}{\alpha^2} = 0$$

$$\Rightarrow \lambda = \frac{1}{2}\left(U \pm \sqrt{U^2 - 4\frac{\hat{k}^2}{\alpha^2}}\right)$$

or

$$\boxed{\lambda_{1,2} = \frac{1}{2}\left(U \pm i\sqrt{4\frac{\hat{k}^2}{\alpha^2} - U^2}\right)}, \text{ let } \omega_k \equiv \sqrt{4\frac{k^2}{\alpha^2} - U^2}$$

$\Rightarrow$ divide $\hat{k}$ to three cases: $\hat{k}_1, \hat{k}_2, \hat{k}_3$:

②

So if $\frac{4\hat{k}_1^2}{\alpha^2} > U^2$, $X = e^{\frac{1}{2}Ux} \cdot \left(E\cos(\omega_k x) + F\sin(\omega_k x)\right)$

if $\frac{4\hat{k}_2^2}{\alpha^2} = U^2$: $X = Ge^{\frac{1}{2}Ux} + He^{\frac{1}{2}Ux} \cdot x$

if $\frac{4\hat{k}_3^2}{\alpha^2} < U^2$: $X = e^{\frac{1}{2}Ux}\left(I\cosh\omega_k x + J\sinh(\omega_k x)\right)$

apply b.c of $X(x) = 0$ for $x = 0, L$. first $x=0$:

$\Rightarrow E = G = H = I = J = 0$.

$\Rightarrow X(x) = Fe^{\frac{1}{2}Ux} \cdot \sin(\omega_k \cdot x)$

then $x = L$: $Fe^{\frac{1}{2}UL}\sin\omega_k L = 0 \Rightarrow \omega_k L = n\pi$

$\Rightarrow \sqrt{4\frac{\hat{k}^2}{\alpha^2} - U^2} \cdot L = n\pi \Rightarrow \frac{4\hat{k}^2}{\alpha^2} = \frac{n^2\pi^2}{L^2} + U^2$

$\boxed{k_n^2 - \gamma = \hat{k}_n^2 = \frac{\alpha^2}{4}\left(\frac{n^2\pi^2}{L^2} + U^2\right)}$. negative $\hat{k}$ may be ignored, they don't add an indep solution.

which n?

those that satisfy

$\boxed{\frac{4k_n^2}{\alpha^2} > U^2.}$, call smallest n: n_0

apply b.c. at $y = L$: $Y'(y=L) = 0$

$\Rightarrow$ easier to write the solution as

$Y = A\cosh\frac{k}{\alpha}(y-L) + B\sinh\frac{k}{\alpha}(y-L) + C(y-L) + D$

③

then apply b.c. at $y=L$ to find:

$$C=B=0 \Rightarrow Y = A\cosh\frac{k}{\alpha}(y-L)+D$$

so solution before applying b.c. at $y=0$ is

$$\bar{u}(x,y) = \sum_{n=n_0}^{\infty} F_n e^{\frac{1}{2}Ux} \cdot \sin\left(\frac{n\pi x}{L}\right) \cdot \cosh\left[\frac{k_n}{\alpha}(y-L)\right]$$

at $y=0$:

$$\bar{u}(x, y=0) = 1 = \sum_{n=n_0}^{\infty} F_n e^{\frac{1}{2}U\cdot x} \sin\frac{n\pi x}{L} \cdot \cosh\left(\frac{k_n}{\alpha}L\right)$$

write this as $1 = \sum_{n=n_0}^{\infty} \hat{F}_n e^{\frac{1}{2}Ux} \cdot \sin\frac{n\pi x}{L}$

where $\hat{F}_n = F_n \cdot \cosh\left(\frac{k_n}{\alpha}L\right)$. to find $\hat{F}_n$, need to write $X(x)$ eq'n in S-L form: use integrating factor: $\sigma(x)$:

$$\sigma X'' - \sigma U X' + \sigma \frac{\hat{k}^2}{\alpha^2} X = 0$$

$$\Rightarrow \sigma'(x) = -\sigma U \Rightarrow \sigma(x) = e^{-Ux}$$

$$\rightarrow \left(e^{-Ux}X'\right)' + e^{-Ux} \cdot \frac{\hat{k}^2}{\alpha^2} X = 0$$

k^2 is the eigenvalue,

$w(x) = \exp(-Ux)/\alpha^2$

$p(x) = e^{-Ux}$

& we know already that $\phi_n = e^{\frac{1}{2}Ux} \cdot \sin\frac{n\pi x}{L}$

so any function may be expressed as a sum over ϕ_n, & $\hat{F}_n$ are given by

$$\hat{F}_n = \frac{\langle 1, \phi_n \rangle}{\langle \phi_n \phi_n \rangle} = \frac{\int_0^L \underbrace{e^{\frac{1}{2}Ux} \sin\frac{n\pi x}{L}}_{\phi_n} \cdot 1 \cdot \underbrace{e^{-Ux}}_{w} \cdot dx}{\int_0^L \left[e^{\frac{1}{2}Ux} \sin\frac{n\pi x}{L}\right]^2 \cdot e^{-Ux}\, dx}$$

$\Rightarrow$ done calculating $\bar{u}(x,y)$. now $v(x,y,t)$.

$v = X \cdot Y \cdot T$

$$\Rightarrow \underbrace{\left(\frac{T'}{T} - \gamma\right)}_{=-(a^2+b^2)} + \underbrace{\left(\hat{U}\frac{X'}{X} - \alpha^2 \frac{X''}{X}\right)}_{=a^2} - \underbrace{\alpha^2 \frac{Y''}{Y}}_{=b^2} = 0$$

$v(x,y,t)$ satisfies homog b.c. on v or v_y

which allow us to eliminate several parts of the general solutions for X & Y, leaving only:

$$X(x) = Ae^{\frac{1}{2}Ux} \sin\frac{n\pi x}{L} \quad , n > n_0 \text{ as before.}$$

Similarly, $Y(y) = B \cos\frac{b}{\alpha}(y-L)$ (satisfies $Y'(y=L)=0$)

& applying $Y(y=0)=0$, we find

$$\cos\frac{b}{\alpha}L = 0 \Rightarrow (bL/\alpha) = \frac{\pi}{2} + m\pi$$

$$\Rightarrow \boxed{b_m^2 = \left(\frac{\pi}{2} + m\pi\right)^2 \cdot \frac{\alpha^2}{L^2}} \quad m = 0,1,2\ldots$$

So we so far have $X = A\phi_n(x)$

$Y = B\,\psi_m(y)$

$$\phi_n = e^{\frac{1}{2}Ux} \sin\frac{n\pi x}{L} \quad , \psi_m = \cos\left[\frac{\left(\frac{\pi}{2}+m\pi\right)}{L}(y-L)\right]$$

the separation constants are given by

$$b_m^2 = \left(\frac{\pi}{2}+m\pi\right)^2 \frac{\alpha^2}{L^2}$$

$$a_n^2 = \frac{\alpha^2}{4}\left(n^2\pi^2+U^2\right)$$

and $T(t) = e^{-(\gamma+a_n^2+b_m^2)t}$. putting all together:

$u(x,y,t) = \bar{u}(x,y) + v(x,y,t)$. so i.c. for

v are $v(x,y,t=0) = f(x,y) - \bar{u}(x,y) \equiv g(x,y)$.

to satisfy i.c. write ~~full~~ solution for v:

$$v = \sum_{n=n_0}^{\infty} \sum_{m=0}^{\infty} A_{mn} \cdot \phi_n(x) \cdot \psi_m(y) \cdot e^{-(\gamma+a_n^2+b_m^2)t}.$$

at $t=0$: $g(x,y) = \sum_n \sum_m A_{mn} \phi_n(x) \cdot \psi_m(y)$.

to find A_{mn} multiply by $\phi_i(x)\cdot w(x)$ and by $\psi_j(y)$ [$w=1$ in this case] and use orthogonality of the S-L eigen functions; also divide by the normalization factors:

$$\int \phi_i^2 w\,dx \,,\ \int \psi_j^2 dy.$$

$$\left\{\int_0^L dy \int_0^L dx \left[g(x,y)\cdot\phi_i(x)\cdot w(x)\cdot \psi_j(y)\right]\right\}\cdot\left\{\left(\int_0^L \psi_j^2\, dy\right)\left(\int_0^L \phi_i^2\, w\, dx\right)\right\}^{-1}$$

$$= \sum_{m,n} A_{mn} \cdot \underbrace{\left[\frac{\int_0^L \phi_n(x)\,\phi_i(x)\,w(x)\,dx}{\int_0^L (\phi_i(x))^2 w(x)\,dx}\right]}_{= \delta_{ni}} \underbrace{\left[\frac{\int_0^L \psi_m(y)\cdot\psi_j(y)\,dy}{\int_0^L \psi_j^2(y)\,dy}\right]}_{= \delta_{mj}}$$

$$= A_{ij}$$

So we have the coeffs in the expansion of the i.c.

We now solved for both $\bar{u}(x,y)$ & $v(x,y,t)$. can add them together;

Summary of solution:

$$u(x,y,t) = \bar{u}(x,y) + v(x,y,t).$$

$$\bar{u}(x,y) = \sum_{n=n_0}^{\infty} F_n \cdot \phi_n(x) \cdot \cosh\left[\frac{\kappa_n}{\alpha}(y-L)\right]$$

$$F_n = \int_0^L \phi_n(x) \cdot 1 \cdot w(x)\,dx \Big/ \left[\int_0^L (\phi_n(x))^2 w(x)\,dx\right]$$

$$\phi_n = e^{\frac{1}{2}Ux} \sin\frac{n\pi x}{L} \; ; \quad w(x) = e^{-Ux} \; ; \quad \hat{\kappa}_n^2 = \frac{\alpha^2}{4}(n^2\pi^2 + U^2)$$

n_0 is the smallest one satisfying $\frac{4\hat{\kappa}_n^2}{\alpha^2} > U^2$.

$\hat{\kappa}_n^2 \equiv \kappa_n^2 - \gamma$

$$v(x,y,t) = \sum_{n=n_0}^{\infty} \sum_{m=0}^{\infty} A_{mn}\, \phi_n(x)\, \psi_m(y)\, e^{-(\gamma + a_n^2 + b_m^2)t}$$

$$a_n^2 = \frac{\alpha^2}{4}\left(\frac{n^2\pi^2}{L^2} + U^2\right), \quad b_m^2 = \frac{\alpha^2}{L^2}\left(\frac{\pi}{2} + m\pi\right)^2$$

$$A_{ij} = \frac{\int_0^L dy \int_0^L dx \left[g(x,y) \cdot \phi_i(x) \cdot w(x) \cdot \psi_j(y)\right]}{\left(\int_0^L (\phi_i(x))^2 w(x)\,dx\right) \int_0^L (\psi_j(y))^2\,dy}$$

Integral constraints for steady part $\bar{u}(x,y)$

integrate $\left\{ \hat{U}\,\bar{u}_x = \alpha^2(\bar{u}_{xx} + \bar{u}_{yy}) - \gamma u \right\}$ over $\int_0^L dx \int_0^L dy$

$$\Rightarrow \int dy\, \hat{U}\cdot u\Big|_{x=0}^{L} - \alpha^2 \int dy\, u_x\Big|_{x=0}^{L} - \alpha^2 \int dx\, u_y\Big|_{y=0}^{L} = 0$$

or: $-\int_0^L dy\left[\hat{U}u - \alpha^2 u_x\right]_{x=0} + \int_0^L dy\left[\hat{U}u - \alpha^2 u_x\right]_{x=L}$

advection into left boundary

diffusion into left bndry

same, right bndry.

$$-\int_0^L \alpha^2 u_y\Big|_{y=L} dx + \int_0^L \alpha^2 u_y\, dx = \int_0^L dx \int_0^L dy\,(-\gamma u)$$

diffusion into upper boundary

diffusion into lower boundary

cooling to air above the domain

net flux into the domain must vanish for u (temperature...) there to be at a steady state.

or: advection & diffusion from boundaries balance cooling from interior.