Applied Mathematics 105b:
Ordinary and Partial Differential Equations

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Vector calculus syllabus – long version, will not be covered in class, students expected to have seen this in APM21a,b

1. VECTOR CALCULUS AND TENSORS. downloads
   (a) Motivation: bees do it? yes, bees are using vector calculus.
   (b) Introduction and review: A quick reminder
      i. Scalars, vectors, vector addition, scalar multiplication between a vector and a scalar (**Gr**§9.2, pp 412-414)
      ii. Two and three dimensional scalar fields (temperature) and vector fields (wind, heat flux).
      iii. Dot and cross products (**Gr**§14.2, pp 683-686)
      iv. Cartesian coordinates (**Gr**§14.3 to end of example 1, pp 687-690)
   (c) Einstein index notation and friends:
      i. Sum over repeated indices
      ii. Kronecker delta tensor (**Gr** p 427, eqn 22)
      iii. Levi-Civita tensor/ permutation symbol, cross produces, relation to Kronecker delta, to determinants, misc identities notes.
   (d) Vector calculus: div $\nabla\cdot\vec{a}$, grad $\nabla\phi$, curl $\nabla\times\vec{a}$.
      i. Divergence: definition using a general closed-surface integral over a vector field at the limit of the surface becoming infinitesimal. Derivation for a cube-like surface, and the differential operator. (**Gr**§16.3, pp 761-765, including example 1).
      ii. Gradient: definition, input and output of div, grad, curl; directional derivative (**Gr**§16.4, pp 766-769 until but not including example 3).
      iii. Providing physical intuition for divergence: mass conservation for an incompressible fluid: $\nabla\cdot\mathbf{u}=0$ (**Gr** p 797-799, example 2).
      iv. Providing physical intuition for grad and div: temperature field $T(x,y,x)$, diffusive heat flux vector field $k\nabla T=k(T_x,T_y,T_z)$, diffusive local heating rate given by Laplacian, $div(grad(T))=\nabla^2T=T_{xx}+T_{yy}+T_{zz}$ (derive this). vector_calculus_preliminaries.m;
      v. Curl: definition as a cross product between ∇ and a vector field; physical interpretation via the vorticity of a flow field (**Gr**§16.5 p 774-775 to equation 4; for vorticity interpretation, use section 1 of notes-curl-and-vorticity.pdf).
      vi. Some vector identities (**Gr**§16.6, pp 778-780, until just before example 1; prove equations 4, 12, 13).

vii. Application of curl: Ekman transport, coastal upwelling and fisheries notes-curl-and-coastal-upwelling.pdf).

(e) Integral theorems: divergence, Stokes, Green's, potential of a vector field

i. Divergence (Gauss) theorem: theorem 16.8.1 $\int_V \nabla \cdot \vec{v} dV = \int_S \hat{n} \cdot \vec{v} dA$,

A. Proof for a rectangle domain, outline of a proof for general domain, relationship with fundamental theorem of integral calculus (**Gr**§16.8, pp 792-end of first paragraph on p 795).

B. Physical interpretation in terms of heat fluxes over the surface and integral over the local heating rate (example 3, p 799 - equation 40, p 801).

C. 2d version (which will be needed for proving Stokes theorem below): $\int_A \nabla \cdot \vec{v} dV = \int_C \hat{n} \cdot \vec{v} ds$ (equation 47, example 5, p 803).

ii. Stokes theorem: $\int_S \hat{n} \cdot \nabla \times \vec{v} dA = \oint_C \vec{v} \cdot d\vec{R}$.

A. Line integrals, interpretation as the work done by a force vector over a path (definition of line integral from **Gr**§16.9.1, eqns 2,3,4, p 810-811; for interpretation, start with constant force case, example 2 and Figs 177-178 in **Kr** p 373; proceed to variable force case **Kr** second half of p 423).

B. Example: circulation and aerodynamic lift of a wing, connection to Bernoulli law (eqns 15,16, to end of §16.9.1 on p 814, including Fig. 3), class demo of Bernoulli law.

C. Theorem 16.9.1 and proof for a flat (2d) surface (**Gr**§16.9.2, pp 814-815).

D. Geometric intuition (only qualitative): notes-stokes-theorem-intuition.pdf

E. Application 1: potential theory. • Consider first work done by friction as an example of path-dependent work. When is the work performed by a vector force path-independent? • Introduce vector fields that are gradients of a scalar field (**Kr** p 407, the single paragraph above theorem 3). • Three equivalent conditions for line integrals over a vector field to be path-independent: (1) the force may be expressed in terms of a potential, $\vec{F} = \nabla f$ where $f(x,y,x)$ is a scalar function; (2) $curl \vec{F} = \nabla \times \vec{F} = 0$; or (3) line integrals over a closed path vanish. Note that 2 and 3 are equivalent by Stokes theorem! (**Kr**§10.2, from p 426 to the top paragraph on p 429).

F. Application 2: fluid flow around a cylinder - only derivation of Laplace equation for the potential and for the stream function (notes, the solution is given below in the PDEs section).

iii. Green's theorem: $\int_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial Y} \right) dA = \oint_C P\,dx + Q\,dy$

A. This is simply a private case of stokes theorem, **Gr**§16.9.3 pp 818-819, Fig 9.

B. Application: calculating the area of an ellipse (**Kr**§10.4, p 442, example 2);

C. Application: area of a cardioid, (**Kr**§10.4, p 443, example 3). Animation demonstrating the construction of a cardioid from wikipedia.

D. Infinite number of cardioids in the Mandelbrot set: fractal zoom animation from youtube.

(f) Vector calculus in orthogonal curvilinear coordinates

i. Non-Cartesian coordinates

A. Plane polar coordinates, base vectors and expressions for derivatives of the base vectors. (First derivation: **Gr**§14.6.1 p 700-702, eqns 4a,4b,5,10,11,15,16,18a,b; second derivation: eqns 19a,b to end of page 703).

B. Cylindrical coordinates (first paragraph of **Gr**§14.6.2, p 704 and Fig 5 there)

C. Spherical coordinates (**Gr**§14.6.3, p 705-706, to end of eqns 28).

ii. ∇ in non Cartesian coordinates

A. Cylindrical coordinates (**Gr**§16.7.1 p 783 to end of p 785)

B. Example: curl of solid body rotation in cylindrical coordinates (section 2 of notes_curl_and_vorticity.pdf).

C. Spherical coordinates (**Gr**§16.7.2 eqns 27-33), derivations in HW and sections.

D. General curvilinear coordinates (time permitting, **Gr**§16.7.3, p 789-790)