Physical interpretation of curl via the vorticity of fluid flow

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1 Using Cartesian coordinates

Consider a "solid body rotation" flow. E.g., a rotating filled bucket, with the water rotating with the bucket. Suppose the angular velocity is ω, this means that the velocity is cylindrical coordinates is $\mathbf{v} = (v_r, v_\theta, v_z) = (0, r\omega, 0)$. In Cartesian coordinates this is $\mathbf{v} = (v_x, v_y, v_w) = (x\omega, -y\omega, 0)$. (Easiest way to verify this is to consider the angular velocity of a fluid parcel that is on the y axis, and another one that is on the x axis). The curls of this velocity field is

$$\nabla \times \mathbf{v} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\hat{\mathbf{i}} + \left(\frac{\partial v_z}{\partial x} - \frac{\partial v_x}{\partial z}\right)\hat{\mathbf{j}} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{\mathbf{k}} = (0, 0, 2\omega)$$

So the curl of the velocity field is twice the rate of rotation. We therefore conclude that the curl represents the rotation motion of the vector field.

2 Using cylindrical coordinates

In these coordinates,

$$\nabla = \hat{\mathbf{e}}_r \frac{\partial}{\partial r} + \hat{\mathbf{e}}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \hat{\mathbf{e}}_z \frac{\partial}{\partial z}$$

and the curl of a vector is given by

$$\nabla \times \vec{u} = \left(\frac{1}{r}\frac{\partial v_z}{\partial \theta} - \frac{\partial v_\theta}{\partial z}\right) + \left(\frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r}\right) + \frac{1}{r}\left(\frac{\partial (r v_\theta)}{\partial r} - \frac{\partial v_r}{\partial \theta}\right)$$

for the solid body rotation flow above, the only term that does not vanish identically is

$$\frac{1}{r}\frac{\partial (r v_\theta)}{\partial r} = \frac{1}{r}\frac{\partial (r^2 \omega)}{\partial r} = 2\omega.$$