REVIEW

	Input		*Output*
$\operatorname{div}\mathbf{v} = \nabla\cdot\mathbf{v}$:	vector field $\mathbf{v}$	$\rightarrow$	scalar field $\nabla\cdot\mathbf{v}$
$\operatorname{grad} u = \nabla u$:	scalar field u	$\rightarrow$	vector field ∇u
$\operatorname{curl}\mathbf{v} = \nabla\times\mathbf{v}$:	vector field $\mathbf{v}$	$\rightarrow$	vector field $\nabla\times\mathbf{v}$

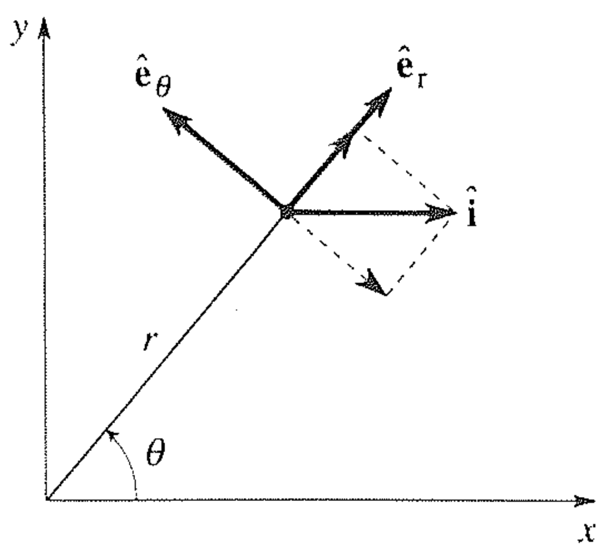
CARTESIAN COORDINATES

$$\boxed{\nabla \equiv \hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z},} \qquad \boxed{\operatorname{grad} u = \frac{\partial u}{\partial x}\hat{\mathbf{i}} + \frac{\partial u}{\partial y}\hat{\mathbf{j}} + \frac{\partial u}{\partial z}\hat{\mathbf{k}}.}$$

$$\begin{aligned}\nabla\cdot\mathrm{v} &= \left(\hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z}\right)\cdot(v_x\hat{\mathbf{i}} + v_y\hat{\mathbf{k}} + v_z\hat{\mathbf{k}}) \\ &= \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z},\end{aligned}$$

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2},$$

POLAR COORDINATES

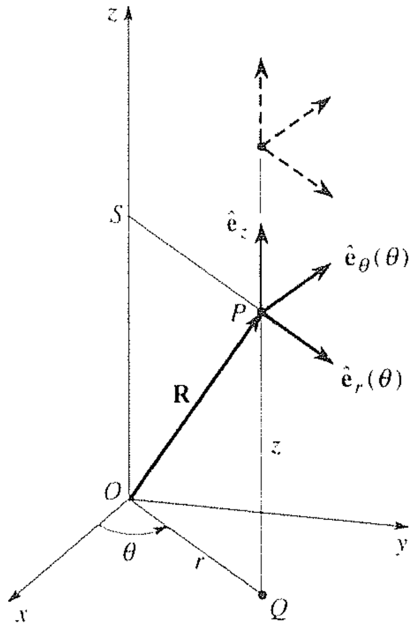


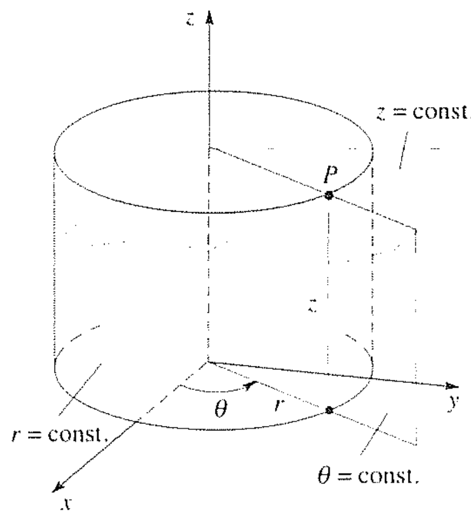
$$x = r\cos\theta, \qquad r = \sqrt{x^2+y^2},$$
$$y = r\sin\theta, \qquad \theta = \tan^{-1}\frac{y}{x},$$

$$\hat{\mathbf{i}} = \cos\theta\,\hat{\mathrm{e}}_r - \sin\theta\,\hat{\mathrm{e}}_\theta, \qquad \hat{\mathrm{e}}_r = \cos\theta\,\hat{\mathbf{i}} + \sin\theta\,\hat{\mathbf{j}},$$
$$\hat{\mathbf{j}} = \sin\theta\,\hat{\mathrm{e}}_r + \cos\theta\,\hat{\mathrm{e}}_\theta. \qquad \hat{\mathrm{e}}_\theta = -\sin\theta\,\hat{\mathbf{i}} + \cos\theta\,\hat{\mathbf{j}}.$$

$$\boxed{\frac{d\hat{\mathrm{e}}_r}{d\theta} = \hat{\mathrm{e}}_\theta \quad \text{and} \quad \frac{d\hat{\mathrm{e}}_\theta}{d\theta} = -\hat{\mathrm{e}}_r.}$$

CYLINDRICAL COORDINATES





$$x = r\cos\theta \qquad r = \sqrt{x^2+y^2},$$
$$y = r\sin\theta \qquad \theta = \tan^{-1}\frac{y}{x},$$
$$z = z$$

$$\hat{\mathbf{i}} = \cos\theta\,\hat{e}_r - \sin\theta\,\hat{e}_\theta,$$
$$\hat{\mathbf{j}} = \sin\theta\,\hat{e}_r + \cos\theta\,\hat{e}_\theta, \qquad \hat{e}_r = \cos\theta\,\hat{\mathbf{i}} + \sin\theta\,\hat{\mathbf{j}},$$
$$\hat{\mathbf{k}} = \hat{e}_z, \qquad \hat{e}_\theta = -\sin\theta\,\hat{\mathbf{i}} + \cos\theta\,\hat{\mathbf{j}}.$$

$$\frac{d\hat{e}_r}{d\theta} = \hat{e}_\theta, \qquad \frac{d\hat{e}_\theta}{d\theta} = -\hat{e}_r$$

$$\boxed{\nabla = \hat{e}_r\frac{\partial}{\partial r} + \hat{e}_\theta\frac{1}{r}\frac{\partial}{\partial\theta} + \hat{e}_z\frac{\partial}{\partial z}.}$$

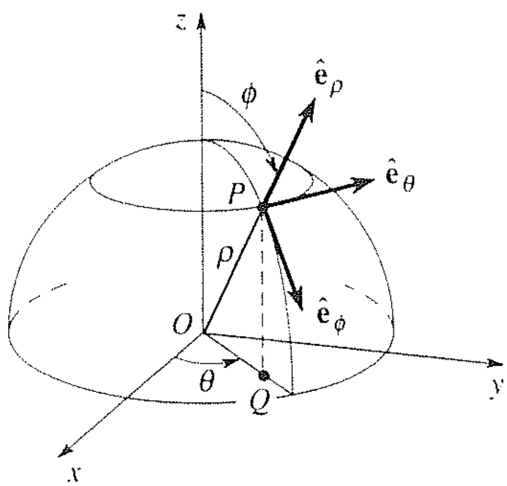
$$\boxed{\nabla u = \frac{\partial u}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial u}{\partial\theta}\hat{e}_\theta + \frac{\partial u}{\partial z}\hat{e}_z,}$$

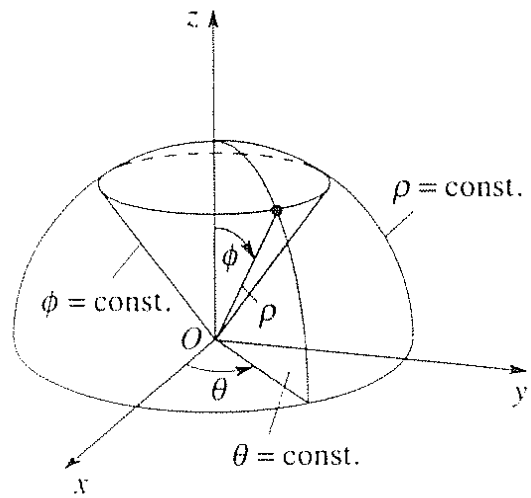
$$\boxed{\nabla\cdot\mathbf{v} = \frac{1}{r}\frac{\partial}{\partial r}(rv_r) + \frac{1}{r}\frac{\partial}{\partial\theta}v_\theta + \frac{\partial}{\partial z}v_z.}$$

$$\boxed{\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2} + \frac{\partial^2 u}{\partial z^2}.}$$

(Polar coordinates are the same, if you leave out the z-terms)

SPHERICAL COORDINATES





$$x = \rho \sin\phi \cos\theta \qquad \rho = \sqrt{x^2 + y^2 + z^2},$$

$$y = \rho \sin\phi \sin\theta \qquad \phi = \cos^{-1} \frac{z}{\sqrt{x^2 + y^2 + z^2}}$$

$$z = \rho \cos\phi, \qquad \theta = \tan^{-1} \frac{y}{x},$$

$$\begin{aligned}
\hat{\mathbf{e}}_\rho &= \sin\phi(\cos\theta \hat{\mathbf{i}} + \sin\theta \hat{\mathbf{j}}) + \cos\phi \hat{\mathbf{k}}, \\
\hat{\mathbf{e}}_\phi &= \cos\phi(\cos\theta \hat{\mathbf{i}} + \sin\theta \hat{\mathbf{j}}) - \sin\phi \hat{\mathbf{k}}, \\
\hat{\mathbf{e}}_\theta &= -\sin\theta \hat{\mathbf{i}} + \cos\theta \hat{\mathbf{j}}.
\end{aligned}$$

$$\begin{aligned}
\hat{\mathbf{i}} &= \sin\phi \cos\theta \hat{\mathbf{e}}_\rho + \cos\phi \cos\theta \hat{\mathbf{e}}_\phi - \sin\theta \hat{\mathbf{e}}_\theta, \\
\hat{\mathbf{j}} &= \sin\phi \sin\theta \hat{\mathbf{e}}_\rho + \cos\phi \sin\theta \hat{\mathbf{e}}_\phi + \cos\theta \hat{\mathbf{e}}_\theta, \\
\hat{\mathbf{k}} &= \cos\phi \hat{\mathbf{e}}_\rho - \sin\phi \hat{\mathbf{e}}_\phi,
\end{aligned}$$

$$\frac{\partial \hat{\mathbf{e}}_\rho}{\partial \rho} = \mathbf{0}, \qquad \frac{\partial \hat{\mathbf{e}}_\rho}{\partial \phi} = \hat{\mathbf{e}}_\phi, \qquad \frac{\partial \hat{\mathbf{e}}_\rho}{\partial \theta} = \sin\phi\, \hat{\mathbf{e}}_\theta,$$

$$\frac{\partial \hat{\mathbf{e}}_\phi}{\partial \rho} = \mathbf{0}, \qquad \frac{\partial \hat{\mathbf{e}}_\phi}{\partial \phi} = -\hat{\mathbf{e}}_\rho, \qquad \frac{\partial \hat{\mathbf{e}}_\phi}{\partial \theta} = \cos\phi\, \hat{\mathbf{e}}_\theta,$$

$$\frac{\partial \hat{\mathbf{e}}_\theta}{\partial \rho} = \mathbf{0}, \qquad \frac{\partial \hat{\mathbf{e}}_\theta}{\partial \phi} = \mathbf{0}, \qquad \frac{\partial \hat{\mathbf{e}}_\theta}{\partial \theta} = -\sin\phi\, \hat{\mathbf{e}}_\rho - \cos\phi\, \hat{\mathbf{e}}_\phi,$$

$$\boldsymbol{\nabla} = \hat{\mathbf{e}}_\rho \frac{\partial}{\partial \rho} + \hat{\mathbf{e}}_\phi \frac{1}{\rho} \frac{\partial}{\partial \phi} + \hat{\mathbf{e}}_\theta \frac{1}{\rho \sin\phi} \frac{\partial}{\partial \theta},$$

$$\boldsymbol{\nabla} u = \frac{\partial u}{\partial \rho} \hat{\mathbf{e}}_\rho + \frac{1}{\rho} \frac{\partial u}{\partial \phi} \hat{\mathbf{e}}_\phi + \frac{1}{\rho \sin\phi} \frac{\partial u}{\partial \theta} \hat{\mathbf{e}}_\theta,$$

$$\boldsymbol{\nabla} \cdot \mathbf{v} = \frac{1}{\rho^2} \frac{\partial}{\partial \rho}(\rho^2 v_\rho) + \frac{1}{\rho \sin\phi} \frac{\partial}{\partial \phi}(v_\phi \sin\phi) + \frac{1}{\rho \sin\phi} \frac{\partial v_\theta}{\partial \theta},$$

$$\nabla^2 u = \frac{1}{\rho^2} \left[\frac{\partial}{\partial \rho} \left(\rho^2 \frac{\partial u}{\partial \rho} \right) + \frac{1}{\sin\phi} \frac{\partial}{\partial \phi} \left(\sin\phi \frac{\partial u}{\partial \phi} \right) + \frac{1}{\sin^2\phi} \frac{\partial^2 u}{\partial \theta^2} \right],$$