An example of S-L with integrating factor, const coeff & $q \neq 0$.

(1) $y'' + 6y' + (11+\lambda)y = 0 \qquad y(0) = y(2) = 0.$

$\sigma y'' + 6\sigma y' + (11+\lambda)\sigma y = 0$

$\Rightarrow \sigma' = 6\sigma \Rightarrow \sigma = e^{6x}.$

(2) $\Rightarrow (\underbrace{e^{6x}}_{p} y')' + \underbrace{11e^{6x}}_{q} y + \lambda \underbrace{e^{6x}}_{w} y = 0$

solving (1): $y = e^{kx} \Rightarrow k^2 + 6k + 11 + \lambda = 0$

$\Rightarrow k = -3 \pm i\sqrt{\lambda+2}$

(re-written from $-3 \pm \sqrt{-\lambda-2}$).

two cases: $\boxed{\text{I}}$ $\lambda \leq -2$: $\Rightarrow k$ is real

$\Rightarrow y = Ae^{k_1 x} + Be^{k_2 x}$

b.c $y(0) = y(2) = 0 \Rightarrow A = B = 0$. not good.

$\boxed{\text{II}}$ $\lambda > -2$: k is complex

$y = Ae^{-3x}\cos(x\sqrt{\lambda+2}) + Be^{-3x}\sin(x\sqrt{\lambda+2}).$

apply b.c: $y(0) = 0 \Rightarrow A = 0.$

$$y(2) = 0 \Rightarrow Be^{-3\cdot 2}\cdot \sin 2\sqrt{\lambda+2} = 0$$

$$\Rightarrow 2\sqrt{\lambda+2} = n\pi \qquad (n \neq 0)$$

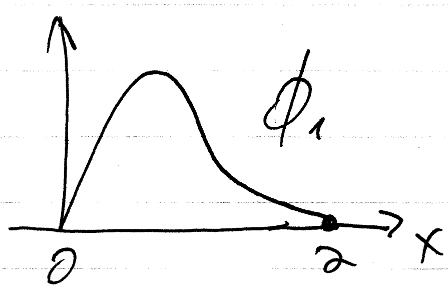
give non-trivial solutions.

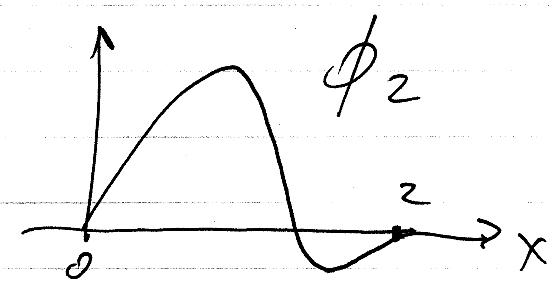
$$\Rightarrow \boxed{\lambda_n = \frac{n^2\pi^2}{4} - 2} \qquad n = 1, 2, 3, \ldots$$

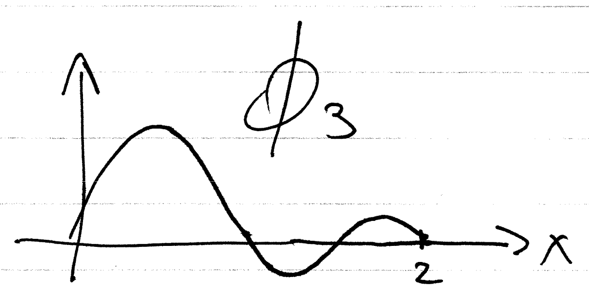
[note: $\lambda_1 = \frac{\pi^2}{4} - 2 = 0.4674$]

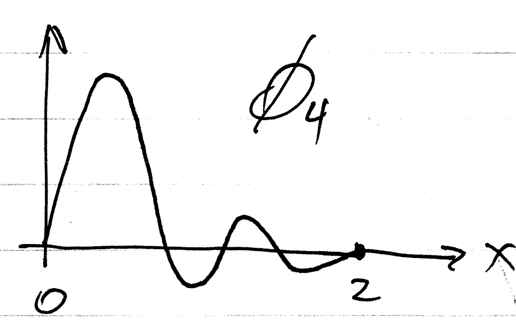
$$\phi_n(x) = Be^{-3x}\sin\left(x\sqrt{\lambda_n+2}\right)$$

$$\Rightarrow \boxed{\phi_n(x) = e^{-3x}\cdot \sin\frac{n\pi x}{2}}$$









* expanding an arbitrary function in terms of $\phi_n(x)$:

let $f(x) = x^2$

$$f(x) = \sum_{n=1}^{\infty} a_n \phi_n(x), \quad a_n = \frac{\langle \phi_n, f(x) \rangle}{\langle \phi_n, \phi_n \rangle}$$

where $\langle \phi_n, f \rangle \equiv \int_0^2 \underbrace{x^2}_{f(x)} \cdot \underbrace{e^{-3x} \sin \frac{n\pi x}{2}}_{\phi_n(x)} \cdot \underbrace{e^{6x}}_{w(x)} dx.$